NAÏVE NONCOMMUTATIVE BLOWING UP

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ABSTRACT. Let $B(X, \mathcal{L}, \sigma)$ be the twisted homogeneous coordinate ring of an irreducible variety X over an algebraically closed field k with $\dim X \geq 2$. Assume that $c \in X$ and $\sigma \in \mathrm{Aut}(X)$ are in sufficiently general position. We show that if one follows the commutative prescription for blowing up X at c, but in this noncommutative setting, one obtains a noncommutative ring $R = R(X, c, \mathcal{L}, \sigma)$ with surprising properties. In particular:

(1) R is always noetherian but never strongly noetherian.
(2) If R is generated in degree one then the images of the R-point modules in qgr-R are naturally in (1-1) correspondence with the closed points of X. However, both in qgr-R and in gr-R, the R-point modules are not parametrized by a projective scheme.
(3) qgr-R has finite cohomological dimension yet $\dim_k \mathrm{H}^1(\mathcal{O}_R) = \infty$.

This gives a more geometric approach to results of the second author who proved similar results for $X = \mathbb{P}^n$ by algebraic methods.

CONTENTS

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1. Introduction

Noncommutative projective geometry has been very successful in using the techniques and intuition of classical algebraic geometry to understand noncommutative connected graded algebras $R = k \oplus \bigoplus_{n\geq 1} R_n$, over an algebraically closed field k. In this paper we show that one of the simplest noncommutative analogues of blowing up a commutative variety leads to algebras with a range of interesting properties.

One reason why classical techniques work is that one can construct nontrivial noncommutative graded rings from commutative data. A typical example is the following. Fix an automorphism σ of a projective k-scheme X and for an invertible sheaf $\mathcal{L}$ write $\mathcal{L}_n = \mathcal{L} \otimes_{\mathcal{O}_X} \sigma^*\mathcal{L} \otimes_{\mathcal{O}_X} \cdots \otimes (\sigma^{n-1})^*\mathcal{L}$, with $\mathcal{L}_0 = \mathcal{O}_X$. Define the *bimodule algebra* $\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma) = \bigoplus_{n\geq 0} \mathcal{L}_n$ with global sections the *twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \sigma) = \bigoplus_{n\geq 0} \mathrm{H}^0(X, \mathcal{L}_n)$. If $\mathcal{L}$ is a σ-ample invertible sheaf, as defined in (2.13) below, then B is a noetherian ring and the category qgr-B of finitely generated right B-modules modulo torsion is equivalent to both qgr-$\mathcal{B}$ and to $\operatorname{coh} X$, the category of coherent sheaves on X.

In many important examples (for example, domains of Gelfand-Kirillov dimension two generated in degree one [AS], Artin-Schelter regular rings of dimension three [ATV] and various higher dimensional algebras) a connected graded k-algebra A has such a ring $B(X, \mathcal{L}, \sigma)$ as a factor. One can then use the geometry of X to understand A and show that it has very pleasant properties. The reason this works is that the *point modules* over A, cyclic A-modules $M = \bigoplus_{i\geq 0} M_i$ with $\dim_k M_i = 1$ for all $i \geq 0$, are parametrized by the scheme X. Combined with the fact that the shift functor $\sigma : M \mapsto M[1]_{\geq 0}$ induces an automorphism on the isomorphism classes of A-modules, this quickly leads to the construction of the factor ring $B(X, \mathcal{L}, \sigma)$. It is important to understand how generally this technique applies. Specifically, it had been hoped that these and related geometric techniques would lead to a classification of noncommutative surfaces, by which we mean qgr-A for a connected graded algebra A with Gelfand-Kirillov dimension three. A general survey of this programme can be found in [SV].

Recently the second author [Ro] constructed examples of noncommutative surfaces that do not have such pleasant properties. For example, although these algebras R are noetherian, they are never *strongly noetherian* (in other words there exists a commutative noetherian k-algebra C such that $R \otimes_k C$ is not noetherian) and their point modules do not appear to be parametrized by a projective scheme. The methods of [Ro] are algebraic. The main aim of this paper is to give an alternative, geometric construction of these algebras that works more generally and helps explain their properties.

Our construction uses a natural noncommutative analogue of blowing up a closed point c on an irreducible variety X. To set this in context, consider first the classical approach. Write $\mathcal{I} = \mathcal{I}_c \subset \mathcal{O}_X$ for the ideal sheaf corresponding to c, let $\mathcal{L}$ be any invertible sheaf on X and form the sheaf of graded algebras $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{I}\mathcal{L} \oplus \mathcal{I}^2\mathcal{L}^{\otimes 2} \oplus \cdots$. Then qgr-$\mathcal{A} \simeq \operatorname{coh} \widetilde{X}$, where $\widetilde{X}$ denotes the blowup of X at c. The sheaf $\mathcal{L}$ is irrelevant to this construction but will be useful later.

Now consider noncommutative analogues of this construction. Using the definition of $\mathcal{B}$ as a guide, it is natural to twist the summands of $\mathcal{A}$ by powers of σ. Thus, write $\mathcal{I}_n = \mathcal{I} \cdot \sigma^*(\mathcal{I}) \cdots (\sigma^{n-1})^*\mathcal{I}$, set $\mathcal{J}_n = \mathcal{I}_n\mathcal{L}_n$ and consider $\mathcal{R} = \mathcal{R}(X, c, \mathcal{L}, \sigma) = \bigoplus_{n\geq 0} \mathcal{J}_n \subset \mathcal{B}(X, \mathcal{L}, \sigma)$ in the place of $\mathcal{A}$. By analogy with

the classical situation, we define qgr-$\mathcal{R}$ to be *the naïve noncommutative blowup* of X at c. Although the category qgr-$\mathcal{R}$ is independent of $\mathcal{L}$, the algebra of global sections

$$R = R(X, c, \mathcal{L}, \sigma) = \mathrm{H}^0(X, \mathcal{R}) = \bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{J}_n)$$

obviously does depend upon $\mathcal{L}$. When $\mathcal{L}$ is very ample, this will be the algebra that interests us.

Before stating the main result, we need one more definition. If $c \in X$ is a closed point, the orbit $\mathcal{C} = \{\sigma^n(c) : n \in \mathbb{Z}\}$ is defined to be *critically dense* if the Zariski closure of any infinite subset $\mathcal{C}'$ of $\mathcal{C}$ equals X. One can regard this as saying that (c, σ) is in sufficiently general position (see Section 11 or [Ro, Theorems 14.5 and 14.6]). Under this hypothesis, R has a range of surprising properties:

Theorem 1.1. *Let X be an irreducible variety with $\dim X \geq 2$ and $\sigma \in \mathrm{Aut}(X)$. Assume that $\mathcal{L}$ is a very ample, σ-ample invertible sheaf on X and that $c \in X$ is a closed point such that $\mathcal{C} = \{\sigma^n(c) : n \in \mathbb{Z}\}$ is critically dense.*

If $\mathcal{R} = \mathcal{R}(X, c, \mathcal{L}, \sigma)$ with global sections $R = R(X, c, \mathcal{L}, \sigma)$, then:

(1) qgr-$R \simeq$ qgr-$\mathcal{R}$ *and* qgr-$\mathcal{R}$ *is independent of the choice of $\mathcal{L}$.*
(2) *R is always noetherian.*
(3) *R is never strongly noetherian.*
(4) *The simple objects in* qgr-R *are in (1-1) correspondence with the closed points of X.*
(5) *When R is generated in degree one, the simple objects in* qgr-R *are the images of the R-point modules. However, in both* qgr-R *and* gr-R*, the R-point modules are not parametrized by any scheme of locally finite type.*
(6) qgr-R *has finite cohomological dimension. When X is smooth,* qgr-R *has finite homological dimension.*
(7) *If $\mathrm{H}^1(R) = \mathrm{Ext}^1_{\text{qgr-}R}(R, R)$, then $\dim_k \mathrm{H}^1(R) = \infty$.*
(8) *R satisfies χ_1 but does not satisfy χ_2, as defined below.*

This theorem summarizes most of our results and so its proof takes up most of the paper. Specifically, parts 1 and 2 of the theorem are proved in Theorem 4.1 and their proof takes up most of Sections 2–4. The rest of the paper is then concerned with applying this theorem to get a deeper understanding of the properties of $\mathcal{R}$ and R. In particular, part3 of Theorem 1.1 is proved in Theorems 9.2 and 9.6; part 4 in Theorem 6.7; part 5 in Theorem 10.4, Corollary 10.5 and Remark 10.8; part 6 in Theorem 8.2 and Corollary 8.3; and parts 7 and 8 in Theorem 7.1.

In the special case where $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}(1)$, most parts of the theorem were proved by more algebraic methods in [Ro], which in turn was motivated by Jordan's work on algebras generated by Eulerian derivatives [Jo]. The significance of [Ro] was to give counterexamples to a number of open problems from the literature and Theorem 1.1 obviously gives further examples. More significantly, it shows that these examples are to be expected within the geometric framework of noncommutative geometry and perhaps also suggests a way of coping with them within, say, the classification of noncommutative surfaces: if one regards these examples as a form of noncommutative blowup then one may hope to classify such algebras using noncommutative analogues of blowing up and down.

Let us explain the significance of individual parts of Theorem 1.1. Part 1 justifies the idea that $\mathcal{R}$ is kind of a noncommutative blowup of X at c. We begin by

discussing this aspect of the theorem since it illustrates the ways in which qgr-$\mathcal{R}$ is both similar and dissimilar to $\operatorname{coh}\widetilde{X}$. In the commutative case, if $\mathcal{L} = \mathcal{O}_X$, then $\mathcal{I}_c\mathcal{A}$ corresponds to an invertible sheaf on $\widetilde{X}$ and the point c is the image of the exceptional divisor $\mathcal{A}/\mathcal{I}_c\mathcal{A} \in \operatorname{coh}\widetilde{X}$. In contrast, an easy computation (Proposition 5.3) shows that $\mathcal{R}/\mathcal{I}_c\mathcal{R}$ is a (finite direct sum of copies) of a simple object $\widetilde{c} \in$ qgr-$\mathcal{R}$. This can be used to prove part 4 of the theorem. Ironically, $\widetilde{c}$ does have some properties that are more like a divisor than a point; in particular $\mathcal{I}_c\mathcal{R}$ is still an invertible module in qgr-$\mathcal{R}$ (see Proposition 5.7). Moreover, if one quotients out the Serre subcategories $\mathcal{S}$ generated by the modules corresponding to the points $\sigma^i(c)$ in the two categories then, as should be expected by analogy with blowing up a commutative variety, the quotient categories (qgr-$\mathcal{R}$)$/\mathcal{S}$ and $(\operatorname{coh} X)/\mathcal{S}$ are equivalent (Proposition 6.9). Remarkably, and in marked contrast to the commutative situation, the subcategory of torsion sheaves in $\operatorname{coh} X$ is also equivalent to the corresponding subcategory of qgr-$\mathcal{R}$, the category of Goldie torsion modules (see Theorem 6.7). Thus, in some respects, the difference between qgr-$\mathcal{R}$ and $\operatorname{coh} X$ is quite subtle.

There is another version [VB2] of noncommutative blowing up that has properties much closer to the classical case and has been useful in describing noncommutative surfaces (see, for example, [SV, Section 13]). Van den Bergh's construction is discussed briefly in Section 5.

The idea of considering strongly noetherian algebras arises in the work of Artin, Small and Zhang [ASZ, AZ2] who show that many algebras have this property, and that it has a number of important consequences for an algebra. Notably, a strongly noetherian graded k-algebra A satisfies *generic flatness* in the following sense: for any finitely generated commutative k-algebra C and any finitely generated $A\otimes_k C$-module M there exists $f \in C \smallsetminus \{0\}$ such that $M[f^{-1}]$ is a flat $C[f^{-1}]$-module [ASZ, Theorem 0.1]. In contrast, R fails generic flatness in a rather dramatic way:

Proposition 1.2. (*Theorem 9.2*) *If U is* any *open affine subset of X, then generic flatness fails for the finitely generated $R \otimes \mathcal{O}_X(U)$-module $\mathcal{R}(U) = \bigoplus \mathcal{J}_n(U)$.*

As was noted earlier, point modules have frequently been used to understand specific classes of noncommutative algebras and [ASZ, AZ2] use generic flatness to provide strong structure theorems for these modules (among others). In particular, if A is a strongly noetherian graded algebra generated in degree one then the point modules for A, both in gr-A and qgr-A, are naturally parametrized by a projective scheme (see [AZ2, Corollary E4.5], respectively Proposition 10.2 below). Moreover, the shift functor $M \mapsto M[1]_{\geq 0}$ induces an automorphism of this scheme (see Proposition 10.2, again).

All three of these results fail for R (see Section 10). This proves Theorem 1.1(5) and is in marked contrast to part 4 of that result. The reason for the dichotomy is that, if one wants to parametrize the point modules in gr-R or qgr-R then, by definition, one needs to parametrize them simultaneously over all base spaces; that is, over $R_C = R \otimes_k C$ for all commutative k-algebras C. However, Proposition 1.2 can be interpreted as saying that $R_{\mathcal{O}(U)}$ has too few point modules in comparison to its localizations $R_{\mathcal{O}_{X,p}}$ at closed points $p \in X$ for such a parametrization to be possible. (See Theorem 10.4 and Corollary 10.5 for the precise statement.)

The χ conditions in part 8 of Theorem 1.1 are defined as follows. A connected graded algebra A satisfies χ_n if $\dim \operatorname{Ext}^i_{A\text{-Mod}}(k, M) < \infty$ for all finitely generated graded A-modules M and all $i \leq n$. These conditions are central to the interplay

between R and qgr-R as described in [AZ1]. In particular χ_1 ensures that one can (essentially) recover the category of finitely generated graded right R-modules gr-R from qgr-R. The higher χ conditions are related to more subtle properties of qgr-R, especially the behavior of cohomology. In fact the failure of χ_2 for R is immediate from the fact that $\mathrm{H}^1(R)$ is infinite dimensional (Theorem 1.1(7)). This result is in contrast with a basic theorem of Serre: $\mathrm{H}^i(Y, \mathcal{F})$ is finite dimensional for any coherent sheaf $\mathcal{F}$ over a projective variety Y and any $i > 0$ [Ha, Theorem III.5.2]. Analogues of this result have also been basic to much of the theory of noncommutative geometry, as developed for example in [AZ1, AZ2], and so that theory is not available for the study of R.

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2. Definitions and background material

As was mentioned in the introduction, we want to work with bimodule algebras like $\mathcal{B} = \bigoplus \mathcal{L}_n$ and in this section we set up the appropriate notation. Most of this comes from [AV] and [VB1] and the reader is referred to those papers for further details.

Fix throughout an integral projective scheme X over an algebraically closed field k. The category of quasicoherent, respectively coherent, sheaves on X will be written $\mathcal{O}_X$-Mod, respectively $\mathcal{O}_X$-mod. We use the following notation for pullbacks: if $\sigma \in \mathrm{Aut}(X)$ is a k-automorphism of X, and $\mathcal{F} \in \mathcal{O}_X$-mod, then $\mathcal{F}^\sigma = \sigma^*(\mathcal{F})$. We adopt the usual convention that an automorphism σ acts on functions by $f^\sigma(x) = f(\sigma(x))$, for $x \in X$.

Definition 2.1. A *coherent $\mathcal{O}_X$-bimodule* is a coherent sheaf $\mathcal{F}$ on $X \times X$ such that $Z = \operatorname{supp} \mathcal{F}$ has the property that both projections $\rho_1, \rho_2 : Z \to X$ are finite morphisms. An *$\mathcal{O}_X$-bimodule* is a sheaf $\mathcal{F}$ on $X \times X$ such that every coherent $X \times X$-subsheaf is a coherent $\mathcal{O}_X$-bimodule. The left and right $\mathcal{O}_X$-module structures associated to $\mathcal{F}$ are defined to be ${}_{\mathcal{O}_X}\mathcal{F} = (\rho_1)_*\mathcal{F}$ and $\mathcal{F}_{\mathcal{O}_X} = (\rho_2)_*\mathcal{F}$ respectively.

The tensor product of two bimodules is again a bimodule and satisfies the expected properties; for example the tensor product is right exact and associative (see [VB1, Section 2]). In fact we will not be concerned with this generality since all the bimodules we consider arise from the following construction.

Definition 2.2. Let $\mathcal{F} \in \mathcal{O}_X$-mod and $\tau, \sigma \in \mathrm{Aut}(X)$. Then define an $\mathcal{O}_X$-bimodule ${}_\tau\mathcal{F}_\sigma$ by $(\tau, \sigma)_*\mathcal{F}$ where $(\tau, \sigma) : X \to X \times X$. We usually write $\mathcal{F}_\sigma$ for ${}_1\mathcal{F}_\sigma$, where 1 is the identity automorphism.

We will see in the next lemma that we only need to consider bimodules of the form ${}_1\mathcal{F}_\sigma$. The reader may check that such a bimodule has left $\mathcal{O}_X$-module structure $\mathcal{F}$ but right $\mathcal{O}_X$-module structure $\mathcal{F}^{\sigma^{-1}}$.

When no other bimodule structure is given, a coherent sheaf $\mathcal{F}$ on X will be assumed to have the bimodule structure ${}_1\mathcal{F}_1$. Thus all sheaves become bimodules, and all tensor products can be thought of as tensor products of bimodules. In order to remove ambiguity, when thinking of a bimodule $\mathcal{G}$ as a sheaf, we mean the left $\mathcal{O}_X$-module structure of $\mathcal{G}$, unless otherwise stated. In particular, when we write $\mathrm{H}^i(X, \mathcal{G})$ or say that $\mathcal{G}$ is generated by its global sections we are referring to the

left structure of $\mathcal{G}$. We often write $\Gamma(\mathcal{G})$ for $\mathrm{H}^0(X,\mathcal{G})$. Working on the left will have notational advantages, but is otherwise not significant since we have:

Lemma 2.3. *Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules, and σ,τ automorphisms of X.*

(1) ${}_\tau\mathcal{F}_\sigma \cong {}_1(\mathcal{F}^{\tau^{-1}})_{\sigma\tau^{-1}}$.
(2) $\mathcal{F}_\sigma \otimes \mathcal{G}_\tau \cong (\mathcal{F}\otimes\mathcal{G}^\sigma)_{\tau\sigma}$.
(3) *The vector space of global sections of ${}_{\mathcal{O}_X}({}_1\mathcal{G}_\tau)$ is naturally isomorphic to that of $({}_1\mathcal{G}_\tau)_{\mathcal{O}_X}$.*

Proof. (1) This follows from the comments after the diagram [VB1, (2.2)], taking $X=Y=V=V''$.

(2) This is a special case of [VB1, Lemma 2.8(2)], where $X=Y=Z=V=W$.

(3) From the comments before the lemma, the right structure of $\mathcal{G}$ is just $\mathcal{G}^{\tau^{-1}}$ and so this has global sections $\mathrm{H}^0(X,\mathcal{G}^{\tau^{-1}})=\mathrm{H}^0(X,\mathcal{G})^{\tau^{-1}}$. □

We can now define bimodule algebras and their categories of modules. Since we only need a special case of Van den Bergh's bimodule algebras, we will only make our definitions in that special case.

Definition 2.4. An *$\mathcal{O}_X$-bimodule algebra* is an $\mathcal{O}_X$-bimodule $\mathcal{B}$ together with a unit map $1:\mathcal{O}_X\to\mathcal{B}$ and a product map $\mu:\mathcal{B}\otimes\mathcal{B}\to\mathcal{B}$ satisfying the usual axioms. Let $\sigma\in\mathrm{Aut}(X)$. The bimodule algebra $\mathcal{B}$ is called a *graded* $(\mathcal{O}_X,\sigma)$-*bimodule algebra* if:

(1) $\mathcal{B}$ decomposes as a direct sum $\mathcal{B}=\bigoplus_{n\geq 0}\mathcal{B}_n$ of $\mathcal{O}_X$-bimodules $\mathcal{B}_n\cong {}_1(\mathcal{E}_n)_{\sigma^n}$, for some $\mathcal{E}_n\in \mathcal{O}_X\text{-mod}$ with $\mathcal{B}_0={}_1(\mathcal{O}_X)_1$.
(2) The multiplication map satisfies $\mu(\mathcal{B}_m\otimes\mathcal{B}_n)\subseteq\mathcal{B}_{m+n}$ for all m,n and $1(\mathcal{O}_X)\subseteq\mathcal{B}_0$. Equivalently μ is defined by $\mathcal{O}_X$-module maps $\mathcal{E}_n\otimes\mathcal{E}_m^{\sigma^n}\to\mathcal{E}_{m+n}$ satisfying the appropriate associativity conditions.

We will write $\mathcal{B}=\bigoplus {}_1(\mathcal{E}_n)_{\sigma^n}$ throughout the section.

Definition 2.5. Let $\mathcal{B}$ be a graded $(\mathcal{O}_X,\sigma)$-algebra. A *right $\mathcal{B}$-module* $\mathcal{M}$ is a quasi-coherent right $\mathcal{O}_X$-module together with a right $\mathcal{O}_X$-module map $\mu:\mathcal{M}\otimes\mathcal{B}\to\mathcal{M}$ satisfying the usual axioms. The module $\mathcal{M}$ is *graded* if $\mathcal{M}=\bigoplus_{n\in\mathbb{Z}}\mathcal{M}_n$ with $\mu(\mathcal{M}_n\otimes\mathcal{B}_m)\subseteq\mathcal{M}_{m+n}$. The *shift of* $\mathcal{M}$ is defined by $\mathcal{M}[n]=\bigoplus\mathcal{M}[n]_i$ with $\mathcal{M}[n]_i=\mathcal{M}_{i+n}$.

The $\mathcal{B}$-module $\mathcal{M}$ is *coherent* (as a $\mathcal{B}$-module) if there is a coherent $\mathcal{O}_X$-module $\mathcal{M}_0$ and a surjective map $\mathcal{M}_0\otimes\mathcal{B}\to\mathcal{M}$ of ungraded $\mathcal{B}$-modules. Left $\mathcal{B}$-modules are defined similarly and the bimodule algebra $\mathcal{B}$ is *right (left) noetherian* if every right (left) ideal of $\mathcal{B}$ is coherent. For the algebras that interest us, a more natural definition of coherence is given in Lemma 3.9.

A priori a graded right $\mathcal{B}$-module $\mathcal{M}=\bigoplus\mathcal{M}_i$ is only a right $\mathcal{O}_X$-module. One can obviously give the $\mathcal{M}_i$ various different bimodule structures and we choose the one that is most convenient. Specifically, it will cause no loss of generality to assume that all right $\mathcal{B}$-modules have the form

$$\mathcal{M}=\bigoplus_{n\in\mathbb{Z}}{}_1(\mathcal{G}_n)_{\sigma^n}\ \text{for some (left) sheaves}\ \mathcal{G}_n\in\mathcal{O}_X\text{-Mod}. \tag{2.6}$$

The advantage of this choice is that the $\mathcal{B}$-module structure on $\mathcal{M}$ is given by a family of $\mathcal{O}_X$-module maps $\mathcal{G}_n\otimes\mathcal{E}_m^{\sigma^n}\to\mathcal{G}_{n+m}$, again satisfying the appropriate associativity conditions.

The graded right $\mathcal{B}$-modules form an abelian category Gr-$\mathcal{B}$, with homomorphisms graded of degree zero. Its subcategory of coherent modules is denoted gr-$\mathcal{B}$. A *bounded* graded $\mathcal{B}$-module $\bigoplus \mathcal{M}_i$ is one such that $\mathcal{M}_i = 0$ for all but finitely many i. A module $\mathcal{M} \in$ Gr-$\mathcal{B}$ is called *torsion* if every coherent submodule of $\mathcal{M}$ is bounded. Let Tors-$\mathcal{B}$ denote the full subcategory of Gr-$\mathcal{B}$ consisting of torsion modules, and write Qgr-$\mathcal{B}$ for the quotient category Gr-$\mathcal{B}$/Tors-$\mathcal{B}$. The analogous quotient category of gr-$\mathcal{B}$ will be denoted qgr-$\mathcal{B}$. The corresponding categories of left modules will be denoted by $\mathcal{B}$-Gr, etc.

Similar category definitions apply to graded rings. A graded k-algebra $A = \bigoplus_{i\geq 0} A_i$ is called *connected graded* if $A_0 = k$. If A is noetherian then a *torsion* right A-module is a graded module such that every finitely generated submodule is bounded. Write Gr-A for the category of graded right A-modules, with torsion subcategory Tors-A and quotient category Gr-A/Tors-A. Similarly, write gr-A for the category of finitely generated right A-modules with quotient category qgr-A = gr-A/tors-A. We denote the natural quotient maps by

$$\pi_{\mathcal{B}} : \text{Gr-}\mathcal{B} \to \text{Qgr-}\mathcal{B} \qquad \text{and} \qquad \pi_A : \text{Gr-}A \to \text{Qgr-}A \tag{2.7}$$

and write both maps as π if no confusion is possible.

We will almost always prove results for right modules over rings or bimodule algebras. By the next lemma, these results will then have a natural counterpart on the left.

Definition 2.8. Let $\psi : X \times X \to X \times X$ be the automorphism given by $(x, y) \mapsto (y, x)$. Let $\mathcal{B}$ be a graded $(\mathcal{O}_X, \sigma)$-algebra. Then the *opposite bimodule algebra* is defined to be $\mathcal{B}^{\mathrm{op}} = \psi_*\mathcal{B}$. The unit and product map for $\mathcal{B}^{\mathrm{op}}$ are induced by ψ_* from the unit and product of $\mathcal{B}$.

Lemma 2.9. *Let $\mathcal{B} = \bigoplus {}_1(\mathcal{E}_n)_{\sigma^n}$ be a graded $(\mathcal{O}_X, \sigma)$-algebra. Then*

$$\mathcal{B}^{op} \cong \bigoplus {}_{\sigma^n}(\mathcal{E}_n)_1 \cong \bigoplus {}_1(\mathcal{E}_n^{\sigma^{-n}})_{\sigma^{-n}}.$$

Thus $\mathcal{B}^{op}$ is a graded $(\mathcal{O}_X, \sigma^{-1})$-algebra. There is a natural category equivalence Gr-$\mathcal{B} \simeq \mathcal{B}^{op}$-Gr *(and similarly for the other module categories).*

Proof. That $\mathcal{B}^{\mathrm{op}} \cong \bigoplus {}_{\sigma^n}(\mathcal{E}_n)_1$ follows immediately from Definition 2.2 while the second isomorphism is just Lemma 2.3(1). If $\psi : X \times X \to X \times X$ is the automorphism given by $(x, y) \mapsto (y, x)$, then ψ_* naturally induces the equivalences of categories. □

The notion of coherence for $\mathcal{B}$-modules should be viewed as an analog of finite generation, but there is a subtlety here: even if $\mathcal{B}$ is right noetherian, it does not seem to follow that every submodule of a coherent $\mathcal{B}$-module is coherent! Fortunately, all the bimodule algebras we need are covered by the following result, so there is no problem.

Proposition 2.10. *Let $\mathcal{B} = \bigoplus \mathcal{B}_i$ be a right noetherian graded $(\mathcal{O}_X, \sigma)$-bimodule algebra and write $\mathcal{B}_i = {}_1(\mathcal{E}_i)_{\sigma^i}$ for each i. Assume that each $\mathcal{E}_i$ is a subsheaf of a locally free sheaf $\mathcal{L}_i$ on X. If $\mathcal{M} \in$ gr-$\mathcal{B}$, then every $\mathcal{B}$-submodule of $\mathcal{M}$ is coherent.*

Thus gr-$\mathcal{B}$ *is an abelian category and a right $\mathcal{B}$-module $\mathcal{N}$ is noetherian if and only if it is coherent.*

Proof. The only step of substance will be to show that submodules of coherent $\mathcal{B}$-modules are coherent. As in the proof of [VB1, Proposition 3.6(3)], let $\mathcal{C}$ denote

the class of $\mathcal{M}_0 \in \mathcal{O}_X$-mod such that every submodule of $\mathcal{M}_0 \otimes \mathcal{B}$ is coherent. We first wish to show that $\mathcal{C} = \mathcal{O}_X$-mod.

Pick a very ample invertible sheaf $\mathcal{O}_X(1)$ over X and a nonzero global section $z \in \mathrm{H}^0(X, \mathcal{O}_X(1))$. Since X is integral, multiplication by z^n, for $n \geq 1$, induces an injection $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(n)$ and hence an injection $\mathcal{O}_X(-n) \otimes \mathcal{L}_i \hookrightarrow \mathcal{L}_i$. As $\mathcal{E}_i \subseteq \mathcal{L}_i$ and $\mathcal{O}_X(-n)$ is locally free, we therefore get an induced embedding $\mathcal{O}_X(-n) \otimes \mathcal{B}_i \hookrightarrow \mathcal{B}_i$. Thus the right $\mathcal{B}$-module morphism $\mathcal{O}_X(-n) \otimes \mathcal{B} \to \mathcal{B}$ induced by (left) multiplication by z^n is an injection for all $n \geq 1$. Therefore, any $\mathcal{B}$-submodule of $\mathcal{O}_X(-n) \otimes \mathcal{B}$ is isomorphic to a right $\mathcal{B}$-ideal and hence is coherent. Hence $\mathcal{O}_X(-n) \in \mathcal{C}$ for all $n \geq 0$.

As in the proof of [VB1, Proposition 3.6(1–2)] it is easy to see that quotients and extensions of coherent right $\mathcal{B}$-modules are coherent and so $\mathcal{C} = \mathcal{O}_X$-mod. By definition, this implies that any $\mathcal{B}$-submodule of a coherent module is coherent. Therefore, the coherent $\mathcal{B}$-modules form an abelian category. Finally, the proof of [VB1, Proposition 3.6(4)] shows that a $\mathcal{B}$-module is noetherian if and only if it is coherent. $\square$

As noted in the introduction we are also interested in the algebra of sections of a bimodule algebra. This is defined as follows. Fix a graded $(\mathcal{O}_X, \sigma)$-algebra $\mathcal{B} = \bigoplus \mathcal{B}_n$ with $\mathcal{B}_n = (\mathcal{E}_n)_{\sigma^n}$ for each n. Then it is clear that the space of (left) global sections $B = \Gamma(\mathcal{B}) = \mathrm{H}^0(X, \mathcal{B})$ has a natural graded k-algebra structure $B = \bigoplus \Gamma(\mathcal{B}_n)$, given by the maps $\Gamma(\mathcal{E}_n) \otimes \Gamma(\mathcal{E}_m^{\sigma^n}) \to \Gamma(\mathcal{E}_{n+m})$ for all $n, m \geq 0$. We call B the *section algebra* of $\mathcal{B}$. Similarly, if $\mathcal{M}$ is a graded right $\mathcal{B}$-module, write $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{G}_n)_{\sigma^n}$ as in (2.6) and define

$$\Gamma(\mathcal{M}) = \bigoplus \Gamma(\mathcal{G}_n) = \bigoplus \mathrm{H}^0(X, \mathcal{G}_n).$$

This is naturally a right B-module, given by maps $\Gamma(\mathcal{G}_n) \otimes \Gamma(\mathcal{E}_m^{\sigma^n}) \to \Gamma(\mathcal{G}_{n+m})$ for all $n \in \mathbb{Z}$, $m \geq 0$. Thus we get a functor $\Gamma : \text{Gr-}\mathcal{B} \to \text{Gr-}B$.

Conversely, for a right B-module M we define $M \otimes_B \mathcal{B}$ to be the sheafification of the presheaf $V \mapsto M \otimes_B \mathcal{B}(V)$ for open $V \subseteq X$. One may check that $M \otimes_B \mathcal{B}$ is naturally a right $\mathcal{B}$-module. Analogously, one may define a left $\mathcal{B}$-module $\mathcal{B} \otimes_B N$ for any left B-module N. The functor $- \otimes \mathcal{B} : \text{Gr-}B \to \text{Gr-}\mathcal{B}$ is a right adjoint to $\Gamma : \text{Gr-}\mathcal{B} \to \text{Gr-}B$.

The next definition and theorem from [VB1] provide an important situation when one can relate the properties of $\mathcal{B}$ and B.

Definition 2.11. Suppose that $\{\mathcal{J}_n\}_{n \in \mathbb{N}}$ is a sequence of $\mathcal{O}_X$-bimodules. Then the sequence is *(right) ample* if, for any $\mathcal{M} \in \mathcal{O}_X$-mod, one has the following:

(1) $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{J}_n$ is generated by global sections for $n \gg 0$.
(2) $\mathrm{H}^i(\mathcal{M} \otimes \mathcal{J}_n) = 0$ for all $i > 0$ and $n \gg 0$.

Theorem 2.12. (Van den Bergh) *Let $\mathcal{B} = \bigoplus \mathcal{B}_i$ be a graded $(\mathcal{O}_X, \sigma)$-algebra. Assume that $\mathcal{B}$ is right noetherian and that $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ is a right ample sequence of $\mathcal{O}_X$-bimodules such that each $\mathcal{B}_n$ is contained in a locally free left $\mathcal{O}_X$-module. Then the section algebra $B = \Gamma(\mathcal{B})$ is right noetherian, and there is an equivalence of categories $\xi : \text{qgr-}\mathcal{B} \simeq \text{qgr-}B$ via the inverse equivalences $\Gamma(-)$ and $- \otimes_B \mathcal{B}$.*

Proof. Since gr-$\mathcal{B}$ is abelian (Proposition 2.10) this follows from the right hand version of [VB1, Theorem 5.2], with one proviso. The conventions for the right handed version of [VB1] require that one defines $M = \Gamma(\mathcal{M})$ for $\mathcal{M} = \bigoplus \mathcal{M}_i \in$

gr-$\mathcal{B}$ by taking the sections of $\mathcal{M}_i$ as a right $\mathcal{O}_X$-module. However, the precise construction of B and M is not important to that proof; all one requires is that the module structure of M is induced from that of $\mathcal{M}$. Thus the proof in [VB1] also proves this theorem.

Alternatively, it is also not hard to check using Lemma 2.3(3) that the left and right section algebras of $\mathcal{B}$ are actually isomorphic as graded rings. $\square$

The first main result of the paper (see Theorem 4.1 or part 1 of Theorem 1.1) will be to show that this theorem can be applied to our naïve blowups. This will then allow us to identify qgr-R with qgr-$\mathcal{R}$ for such algebras.

An important special case of Definition 2.11 and Theorem 2.12 occurs when $\mathcal{J}_n = \mathcal{B}_n = ({}_1\mathcal{L}_\sigma)^{\otimes n}$ for an invertible sheaf $\mathcal{L}$ on X. We will usually write $\mathcal{L}_\sigma^{\otimes n}$ for $({}_1\mathcal{L}_\sigma)^{\otimes n}$. It is customary to say that

(2.13) $\mathcal{L}$ is *σ-ample if* $\{\mathcal{L}_\sigma^{\otimes n}\}_{n\geq 0}$ is a right ample sequence of bimodules.

We will always write $\mathcal{B} = \mathcal{B}(X,\mathcal{L},\sigma) = \bigoplus \mathcal{L}_\sigma^{\otimes n}$ with section algebra

$$B = B(X,\mathcal{L},\sigma) = \bigoplus_{n\geq 0} \Gamma(\mathcal{L}_\sigma^{\otimes n}).$$

This is an equivalent definition of the *twisted homogeneous coordinate ring* of X from the introduction. The σ-ampleness condition is subtle since given a projective scheme X with automorphism σ, there may be no σ-ample sheaves. However, it is known when one such sheaf exists and in that case all ample invertible sheaves are automatically σ-ample, on both the left and the right. For these and further results about $B(X,\mathcal{L},\sigma)$, see [AV, Ke1].

3. Rees bimodule algebras

In this section we formally define the algebras $\mathcal{R} = \mathcal{R}(X,c,\mathcal{L},\sigma)$ and $R = \Gamma(\mathcal{R})$ from Theorem 1.1 and give conditions under which $\mathcal{R}$ is noetherian. In the next section we will consider when the sequence $\{\mathcal{R}_n\}$ is ample in the sense of Definition 2.11. Once this has been done, Van den Bergh's Theorem 2.12 can be applied to show that $\mathcal{R}$ and R are noetherian in considerable generality.

The following assumptions will be fixed throughout the section.

Assumptions 3.1. Fix an integral projective scheme X. Fix $\sigma \in \mathrm{Aut}(X)$, an invertible sheaf $\mathcal{L}$ on X and let $\mathcal{I} = \mathcal{I}_c$ denote the sheaf of ideals defining a closed point c on X. Assume that c has infinite order under σ and write $c_i = \sigma^{-i}(c)$ for $i \in \mathbb{Z}$. Our convention on automorphisms from the beginning of Section 2 means that $\mathcal{I}^{\sigma^i} = \mathcal{I}_{c_i}$ with quotient $\mathcal{O}_X/\mathcal{I}_{c_i} = k(c_i)$ the corresponding skyscraper sheaf.

Mimicking classical blowing up we set

$$\mathcal{I}_n = \mathcal{I}\mathcal{I}^\sigma \dots \mathcal{I}^{\sigma^{n-1}}, \quad \mathcal{L}_n = \mathcal{L}\otimes\mathcal{L}^\sigma\otimes\cdots\otimes\mathcal{L}^{\sigma^{n-1}} \quad \text{and} \quad \mathcal{R}_n = {}_1(\mathcal{I}_n\otimes\mathcal{L}_n)_{\sigma^n},$$

where all tensor products are over $\mathcal{O}_X$. From this data we define a bimodule algebra

$$\mathcal{R} = \mathcal{R}(X,c,\mathcal{L},\sigma) = \bigoplus_{n=0}^{\infty} \mathcal{R}_n$$

with corresponding algebra of sections

$$R = R(X,c,\mathcal{L},\sigma) = \Gamma(\mathcal{R}) = \bigoplus \Gamma(\mathcal{R}_n).$$

Note that by Lemma 2.9,

$$\mathcal{R}(X,c,\mathcal{L},\sigma)^{\mathrm{op}} = \mathcal{R}(X,\sigma(c),\mathcal{L}^{\sigma^{-1}},\sigma^{-1}). \tag{3.2}$$

Thus any results on the right can immediately be transferred to the left.

As the next two easy lemmas show, the hypotheses from (3.1) allow us to replace products by tensor products in the definition of $\mathcal{I}_n$ and so there should be no confusion between this and the definition of $\mathcal{L}_n$.

Lemma 3.3. *For $i = 1, 2$, let $\mathcal{F}_i \in \mathcal{O}_X$-Mod, and let $Z_i \subseteq X$ be the set of closed points of X at which $\mathcal{F}_i$ is not locally free. Then:*

(1) *If $\mathcal{T}or$ denotes sheaf Tor then $\operatorname{supp} \mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{F}_1,\mathcal{F}_2) \subseteq Z_1 \cap Z_2$ for $j > 0$.*

(2) *If $\mathcal{K} \subseteq \mathcal{O}_X$ is an ideal sheaf with $\operatorname{supp}(\mathcal{O}_X/\mathcal{K}) \cap Z_2 = \emptyset$, then $\mathcal{K}\otimes\mathcal{F}_2 \cong \mathcal{K}\mathcal{F}_2$.*

Proof. By [GH, p. 700], it suffices to prove this locally, where it is obvious. $\square$

Lemma 3.4. *For $n \geq 1$, there are natural isomorphisms $\mathcal{I}_n \cong \mathcal{I} \otimes \cdots \otimes \mathcal{I}^{\sigma^{n-1}}$ and $\mathcal{I}_n \otimes \mathcal{L}_n \cong \mathcal{I}_n\mathcal{L}_n$.*

If $\mathcal{R} = \mathcal{R}(X,c,\mathcal{L},\sigma)$, then the natural homomorphism $\mathcal{R}_m \otimes \mathcal{R}_n \to \mathcal{R}_{m+n}$ is an isomorphism for all $m, n \geq 0$. In particular, $\mathcal{R}_n \cong (\mathcal{R}_1)^{\otimes n}$ as $\mathcal{O}_X$-bimodules.

Proof. Since $\mathcal{O}_X/\mathcal{I}^{\sigma^i} \cong k(c_i)$ and the points c_i are distinct, the first result follows from Lemma 3.3(2). The second assertion follows from this together with Lemma 2.3. $\square$

The choice of $\mathcal{L}$ in Assumptions 3.1 is important to the study of the section algebra R but, as in the classical case [Ha, Lemma II.7.9], it is irrelevant to the study of the bimodule algebra $\mathcal{R}$.

Proposition 3.5. *Given two invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$, write $\mathcal{R} = \mathcal{R}(X,c,\mathcal{L},\sigma)$, respectively $\mathcal{R}' = \mathcal{R}(X,c,\mathcal{L}',\sigma)$. Then $\mathcal{R}$-Gr $\simeq \mathcal{R}'$-Gr and Gr-$\mathcal{R} \simeq$ Gr-$\mathcal{R}'$.*

Proof. We prove the result on the right; the left-sided result follows from Lemma 2.9. It also suffices to prove the result when $\mathcal{L}' = \mathcal{O}_X$. As in (2.6), we write an arbitrary module $\mathcal{M}' \in \mathcal{R}'$-Gr as $\mathcal{M}' = \bigoplus {}_1(\mathcal{G}_n)_{\sigma^n}$, so that the module structure is given by homomorphisms of sheaves $\alpha_{n,m} : \mathcal{G}_n \otimes \mathcal{I}_m^{\sigma^n} \to \mathcal{G}_{n+m}$ for all $n \in \mathbb{Z}$ and $m \geq 0$.

We wish to construct a right $\mathcal{R}$-module $\mathcal{M} = \bigoplus {}_1(\mathcal{G}_n \otimes \mathcal{L}_n)_{\sigma^n}$ from $\mathcal{M}'$. To define the module structure requires maps of sheaves

$$\beta_{n,m} : (\mathcal{G}_n \otimes \mathcal{L}_n) \otimes (\mathcal{I}_m \otimes \mathcal{L}_m)^{\sigma^n} \to \mathcal{G}_{n+m} \otimes \mathcal{L}_{n+m}.$$

However, since $\mathcal{L}_n \otimes (\mathcal{L}_m)^{\sigma^n} \cong \mathcal{L}_{m+n}$, the maps $\beta_{n,m}$ arise naturally by tensoring the given maps $\alpha_{n,m}$ with $\mathcal{L}_{m+n}$. Checking that this does indeed define a module structure on $\mathcal{M}$ is straightforward.

This gives a functor $\theta :$ Gr-$\mathcal{R}' \to$ Gr-$\mathcal{R}$. A similar argument will construct a functor $\psi :$ Gr-$\mathcal{R} \to$ Gr-$\mathcal{R}'$ which sends an $\mathcal{R}$-module $\bigoplus {}_1(\mathcal{G}_n)_{\sigma^n}$ to the $\mathcal{R}'$-module $\bigoplus {}_1(\mathcal{G}_n \otimes \mathcal{L}_n^{-1})_{\sigma^n}$. It is obvious that $\psi\theta$ and $\theta\psi$ are naturally isomorphic to identity functors, so that Gr-$\mathcal{R}$ and Gr-$\mathcal{R}'$ are equivalent. $\square$

The main result of this section (Proposition 3.10) determines when the bimodule algebra $\mathcal{R}$ is noetherian. The answer will involve the following geometric notion.

Definition 3.6. Let $\mathcal{C}$ be an infinite set of closed points of an integral scheme X. Then we say that $\mathcal{C}$ is *critically dense* if every infinite subset of $\mathcal{C}$ has Zariski closure equal to X.

We start with some easy observations.

Lemma 3.7. *If $\mathcal{C} = \{c_n : n \in \mathbb{Z}\}$ is critically dense then X is smooth at every point $c_i \in \mathcal{C}$.*

Proof. If some c_i lies in the non-smooth locus Z, then so does every point $c_j = \sigma^{i-j}(c_i)$. Hence the closure X of $\mathcal{C}$ is contained in Z, which is absurd. □

Lemma 3.8. *Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{O}_X$ be comaximal ideal sheaves. Assume that $\mathcal{W}$ is an ideal sheaf such that $\mathcal{U} \cap \mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{V}$. Then $\mathcal{Z} = \mathcal{U} + \mathcal{W}$ is the unique largest sheaf of ideals such that $\mathcal{Z}\mathcal{V} \subseteq \mathcal{W}$. Moreover, $\mathcal{Z}\mathcal{V} = \mathcal{Z} \cap \mathcal{V} = \mathcal{W}$.*

Proof. This follows from a repeated use of comaximality. □

The definition of a coherent $\mathcal{R}$-module from Definition 2.5 is not convenient for most applications and so we will use the following equivalent definition.

Lemma 3.9. *The following are equivalent for a module $\mathcal{M} = \bigoplus_{n\in\mathbb{Z}} \mathcal{M}_n \in \text{Gr-}\mathcal{R}$:*

1. *$\mathcal{M}$ is coherent.*
2. *Each $\mathcal{M}_n$ is a coherent $\mathcal{O}_X$-module, with $\mathcal{M}_n = 0$ for $n \ll 0$, and the natural map $\mu_n : \mathcal{M}_n \otimes \mathcal{R}_1 \to \mathcal{M}_{n+1}$ is surjective for $n \gg 0$.*

Proof. If $\mathcal{M}$ is coherent, there is a surjection $\mathcal{F} \otimes \mathcal{R} \twoheadrightarrow \mathcal{M}$ for some $\mathcal{F} \in \mathcal{O}_X$ -mod. Equivalently, for some $a \le b$ there is a graded surjection $\Theta : \bigoplus_{m=a}^{b} \mathcal{F}_m \otimes \mathcal{R}[-m] \twoheadrightarrow \mathcal{M}$, where each $\mathcal{F}_m$ is a copy of $\mathcal{F}$ situated in degree m. Clearly each $\mathcal{M}_n$ is then a coherent sheaf and $\mathcal{M}_n = 0$ for $n \ll 0$. For $n \ge b$ we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_m \mathcal{F}_m \otimes \mathcal{R}_{n-m} \otimes \mathcal{R}_1 & \xrightarrow{\theta_1} & \mathcal{M}_n \otimes \mathcal{R}_1 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ \bigoplus_m \mathcal{F}_m \otimes \mathcal{R}_{n-m+1} & \xrightarrow{\theta_2} & \mathcal{M}_{n+1} \end{array}$$

where the θ_i are induced from Θ and are therefore surjections, while ϕ_1 is the natural isomorphism. Thus, $\phi_2 \cong \mu_n$ is a surjection.

For the other direction, we may assume that $\mathcal{M} = \bigoplus_{n=a}^{\infty} \mathcal{M}_n$ where each $\mathcal{M}_n$ is coherent, and that μ_n is a surjection for all $n \ge b$. Then $\mathcal{F} = \bigoplus_{m=a}^{b} \mathcal{M}_m$ is a coherent $\mathcal{O}_X$-module and there is a surjective map $\mathcal{F} \otimes \mathcal{R} \to \mathcal{M}$. □

Proposition 3.10. *Keep the hypotheses of* (3.1). *The bimodule algebra $\mathcal{R} = \mathcal{R}(X, c, \mathcal{L}, \sigma)$ is right noetherian if and only if $\{c_i\}_{i\ge 0}$ is a critically dense subset of X, left noetherian if and only if $\{c_i\}_{i<0}$ is critically dense in X, and noetherian if and only if $\{c_i\}_{i\in\mathbb{Z}}$ is critically dense in X.*

When $\mathcal{R}$ is not right noetherian, there exists an infinite ascending chain of coherent right ideals of $\mathcal{R}$ with non-torsion factors.

Proof. By Lemma 3.5, the result is independent of the choice of $\mathcal{L}$ and we choose $\mathcal{L} = \mathcal{O}_X$.

Assume first that $\{c_i\}_{i\ge 0}$ is critically dense in X. We need to show that every right ideal of $\mathcal{R}$ is a coherent $\mathcal{R}$-module in the sense of Definition 2.5. By Lemma 3.4, $\mathcal{R}_m \otimes \mathcal{R}_n \cong \mathcal{R}_m\mathcal{R}_n$ for all $m, n \ge 0$, and so we may use products of bimodules in place of tensor products in the proof. An arbitrary right ideal $\mathcal{G}$ of $\mathcal{R}$ is given by a sequence of bimodules $\mathcal{G}_i = {}_1(\mathcal{H}_i)_{\sigma^i} \subseteq \mathcal{R}_i$ such that $\mathcal{G}_i\mathcal{R}_1 \subseteq \mathcal{G}_{i+1}$ for all $i \ge 0$. By

Lemma 3.9, $\mathcal{G}$ will be coherent if and only if $\mathcal{G}_i\mathcal{R}_1 = \mathcal{G}_{i+1}$ for $i \gg 0$. Equivalently, we are given that

$$\mathcal{H}_i\mathcal{I}^{\sigma^i} \subseteq \mathcal{H}_{i+1} \subseteq \mathcal{I}\mathcal{I}^{\sigma}\cdots\mathcal{I}^{\sigma^i} \qquad \text{for all } i \geq 0, \tag{3.11}$$

and $\mathcal{G}$ is coherent if and only if $\mathcal{H}_i\mathcal{I}^{\sigma^i} = \mathcal{H}_{i+1}$ for $i \gg 0$.

Assume that $\mathcal{G} \neq 0$ and pick r such that $\mathcal{H}_r \neq 0$. By critical density, $\mathcal{H}_r$ can only be contained in finitely many ideals $\mathcal{I}_j$ and so there exists $m \geq r$ such that $\mathcal{H}_r \not\subseteq \mathcal{I}^{\sigma^j} = \mathcal{I}_{c_j}$, for all $j \geq m$. Set $\mathcal{U} = \mathcal{H}_m$ and, for $n > m$, put $\mathcal{W} = \mathcal{H}_n$ and $\mathcal{V} = \mathcal{V}_n = \prod_{i=m}^{n-1}\mathcal{I}_{c_i}$. By the choice of m and (3.11), $\mathcal{U} \not\subseteq \mathcal{I}_{c_i}$ for $i \geq m$ and so $\mathcal{U}$ and $\mathcal{V}_n$ are comaximal. Thus (3.11) and induction implies that $\mathcal{U} \cap \mathcal{V}_n = \mathcal{U}\mathcal{V}_n \subseteq \mathcal{W} \subseteq \mathcal{V}_n$. Thus Lemma 3.8 implies that $\mathcal{Z}_n = \mathcal{H}_m + \mathcal{H}_n$ is maximal with respect to $\mathcal{Z}_n\mathcal{V}_n \subseteq \mathcal{H}_n$. Since

$$\mathcal{Z}_n\mathcal{V}_{n+1} = \mathcal{Z}_n\mathcal{V}_n\mathcal{I}^{\sigma^n} \subseteq \mathcal{H}_n\mathcal{I}^{\sigma^n} \subseteq \mathcal{H}_{n+1},$$

this implies that $\mathcal{Z}_n \subseteq \mathcal{Z}_{n+1}$ for all $n \geq m$.

Thus we may pick $n_0 \geq m$ such that $\mathcal{Z}_n = \mathcal{Z}_{n+1}$ for all $n \geq n_0$. For all such n, Lemma 3.8 implies that $\mathcal{H}_{n+1} = \mathcal{Z}_n\mathcal{V}_{n+1} = \mathcal{Z}_n\mathcal{V}_n\mathcal{I}^{\sigma^n} = \mathcal{H}_n\mathcal{I}^{\sigma^n}$. Thus, $\mathcal{G}$ is coherent and $\mathcal{R}$ is right noetherian.

Conversely, suppose that $\{c_i\}_{i\geq 0}$ is not critically dense. Then there exists an infinite set $A \subseteq \mathbb{N}$ such that the Zariski closure of the set of points $\{c_i\}_{i\in A}$ is a reduced closed subscheme $Y \subsetneq X$ with defining ideal say $\mathcal{I}_Y = \bigcap_{i\in A}\mathcal{I}^{\sigma^i}$. Set $\mathcal{H}_n = \mathcal{I}_Y \cap \mathcal{I}_n$ for $n \geq 0$, and observe that $\mathcal{G} = \bigoplus \mathcal{G}_n = \bigoplus {}_1(\mathcal{H}_n)_{\sigma^n}$ is a right ideal of $\mathcal{R}$. Write $\mathfrak{m}_{c_n}$ for the maximal ideal in the local ring $\mathcal{O}_{X,c_n}$. Looking locally at the point c_n, for $n \in A$, we have $(\mathcal{H}_n\mathcal{I}^{\sigma^n})_{c_n} = (\mathcal{I}_Y)_{c_n}\mathfrak{m}_{c_n}$ while $(\mathcal{H}_{n+1})_{c_n} = (\mathcal{I}_Y)_{c_n}$. But $(\mathcal{I}_Y)_{c_n} \neq 0$ and so, by Nakayama's lemma, $(\mathcal{I}_Y)_{c_n}\mathfrak{m}_{c_n} \neq (\mathcal{I}_Y)_{c_n}$. Thus $\mathcal{H}_n\mathcal{I}^{\sigma^n} \neq \mathcal{H}_{n+1}$ for any $n \in A$. Therefore, the right ideal $\mathcal{G}$ is not coherent and $\mathcal{R}$ fails to be right noetherian.

By (3.2), $\mathcal{R}^{\mathrm{op}} \cong \mathcal{R}(X, \sigma(c), \mathcal{L}^{\sigma^{-1}}, \sigma^{-1})$ and so the result on the left follows from the one on the right. The result for noetherian algebras is then obvious.

To prove the final statement, assume that $\mathcal{R}$ is not right noetherian, and let $\mathcal{G} = \bigoplus \mathcal{G}_n$ be the non-coherent right ideal defined above. Set $\mathcal{M}^j = \sum_{0\leq i\leq j}\mathcal{G}_i\mathcal{R}$; thus $(\mathcal{M}^j)_n = ((\mathcal{I}_Y \cap \mathcal{I}_j)\mathcal{I}^{\sigma^{j+1}}\cdots\mathcal{I}^{\sigma^{n-1}})_{\sigma^n}$ for $n \geq j$. This gives a chain of coherent right ideals $\mathcal{M}^0 \subseteq \mathcal{M}^1 \subseteq \cdots \subset \mathcal{R}$. Looking locally at a point $c_m \in A$ one finds that $\mathcal{M}_i^m \subsetneq \mathcal{M}_i^{m+1}$ for all $m \in A$ and all $i \geq m$. Thus the subsequence $\{\mathcal{M}^m : m \in A\}$ gives the desired ascending chain of right ideals of $\mathcal{R}$. $\square$

4. Ampleness

We maintain the hypotheses from (3.1). The main aim of this section (Theorem 4.1) is to prove, in considerable generality, that the sequence of bimodules $\{\mathcal{R}_n = (\mathcal{I}_n \otimes \mathcal{L}_n)_{\sigma^n}\}$ is ample in the sense of Definition 2.11. Combined with the results of the last two sections this will prove parts 1 and 2 of Theorem 1.1.

Theorem 4.1. *Assume that X is an integral projective scheme such that $X \not\cong \mathbb{P}^1$ and fix $\sigma \in \operatorname{Aut}(X)$. Let $c \in X$ be a closed point for which $\{c_i\}_{i\in\mathbb{Z}}$ is critically dense in X. Suppose that $\mathcal{L}$ is a very ample and σ-ample invertible sheaf on X.*

Set $\mathcal{R} = \mathcal{R}(X, c, \mathcal{L}, \sigma)$. Then the section algebra $R = \Gamma(\mathcal{R})$ is noetherian and there is an equivalence of categories ξ : $\mathrm{qgr}\text{-}\mathcal{R} \simeq \mathrm{qgr}\text{-}R$ via the inverse equivalences $\Gamma(-)$ and $- \otimes_R \mathcal{R}$. Similarly, $\mathcal{R}\text{-}\mathrm{qgr} \simeq R\text{-}\mathrm{qgr}$.

We prove this theorem through a series of reductions that give increasingly simple criteria for the ampleness of the sequence $\{\mathcal{R}_n\}$. Note that only the left sheaf structure of the bimodules $\mathcal{R}_n$ really matters in the definition of right ampleness and we write $\mathcal{J}_n = \mathcal{I}_n \otimes \mathcal{L}_n$ for that left sheaf. By Lemma 3.4 we can and will identify $\mathcal{J}_n$ with $\mathcal{I}_n\mathcal{L}_n \subset \mathcal{L}_n$ for any $n \geq 1$.

Lemma 4.2. *The following are equivalent:*

(1) *The sequence $\{\mathcal{R}_n\}$ is ample.*
(2) *The sheaf $\mathcal{L}$ is σ-ample and $\mathrm{H}^1(\mathcal{M} \otimes \mathcal{R}_n) = 0$ for all invertible sheaves $\mathcal{M}$ and all $n \gg 0$.*

Proof. Suppose that (1) holds and let $\mathcal{M}$ be an arbitrary invertible sheaf. Since $\mathcal{M}$ is flat, the sequence

$$0 \to \mathcal{M} \otimes \mathcal{J}_n \to \mathcal{M} \otimes \mathcal{L}_n \to \mathcal{M} \otimes \mathcal{L}_n/\mathcal{J}_n \to 0$$

is exact. Fix $i > 0$. Since $\mathcal{L}_n/\mathcal{J}_n$ is supported on a finite set of points, so is $\mathcal{M}\otimes\mathcal{L}_n/\mathcal{J}_n$. Therefore, $\mathrm{H}^i(\mathcal{M}\otimes\mathcal{L}_n/\mathcal{J}_n) = 0$ while, by hypothesis, $\mathrm{H}^i(\mathcal{M}\otimes\mathcal{J}_n) = 0$ for $n \gg 0$. Thus $\mathrm{H}^i(\mathcal{M} \otimes \mathcal{L}_n) = 0$ for $n \gg 0$ and so [AV, Proposition 3.4] implies that $\mathcal{L}$ is σ-ample. The claim regarding H^1 is a special case of Definition 2.11(2).

Now suppose that (2) holds and let $\mathcal{F}$ be an arbitrary coherent sheaf. There is an exact sequence

$$0 \to \mathcal{T}or_1(\mathcal{F}, \mathcal{L}_n/\mathcal{J}_n) \to \mathcal{F} \otimes \mathcal{J}_n \to \mathcal{F} \otimes \mathcal{L}_n \to \mathcal{F} \otimes \mathcal{L}_n/\mathcal{J}_n \to 0.$$

By Lemma 3.3(1), $\mathcal{T}or_1(\mathcal{F}, \mathcal{L}_n/\mathcal{J}_n)$ and $\mathcal{F}\otimes\mathcal{L}_n/\mathcal{J}_n$ are supported on finitely many points and hence their higher cohomology vanishes. Thus $H^i(\mathcal{F}\otimes\mathcal{J}_n) \cong H^i(\mathcal{F}\otimes\mathcal{L}_n)$ for $i > 1$. Since $\{\mathcal{L}_n\}$ is an ample sequence, these groups vanish for $n \gg 0$.

It remains to consider $H^1(\mathcal{F} \otimes \mathcal{J}_n)$. There exists a short exact sequence $0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$ where $\mathcal{E}$ is a direct sum of invertible sheaves. Tensoring with $\mathcal{J}_n$ gives the exact sequence

$$0 \to \mathcal{T}or_1(\mathcal{F}, \mathcal{J}_n) \to \mathcal{K} \otimes \mathcal{J}_n \to \mathcal{E} \otimes \mathcal{J}_n \to \mathcal{F} \otimes \mathcal{J}_n \to 0.$$

Once again, $\mathcal{T}or_1(\mathcal{F}, \mathcal{J}_n)$ is finitely supported and so $\mathrm{H}^i(\mathcal{T}or_1(\mathcal{F}, \mathcal{J}_n)) = 0$ for $i > 0$. By the last paragraph, $\mathrm{H}^i(\mathcal{K} \otimes \mathcal{J}_n) = 0$, for $i > 1$ and $n \gg 0$ while, by hypothesis, $\mathrm{H}^1(\mathcal{E} \otimes \mathcal{J}_n) = 0$ for $n \gg 0$. Thus $\mathrm{H}^i(\mathcal{F} \otimes \mathcal{J}_n) = 0$, for $i > 0$ and $n \gg 0$. Hence Definition 2.11(2) holds for $\mathcal{F}$.

Let $\mathcal{O}(1)$ be an arbitrary very ample invertible sheaf on X. Apply the conclusion of the last paragraph to $\mathcal{F} \otimes \mathcal{O}(-i)$, for $1 \leq i \leq \dim X$. This implies that, for some $n_0 \geq 0$, one has

$$\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{O}(-i) \otimes \mathcal{J}_n) = 0 \qquad \text{for } i > 0 \text{ and } n \geq n_0.$$

By [Kl, Proposition 1, p. 307] this implies that $\mathcal{F} \otimes \mathcal{J}_n$ is generated by its sections for all $n \geq n_0$. Thus Definition 2.11(1) holds for $\mathcal{F}$. $\square$

Our second reduction gives a criterion for ampleness based on the following notion of separating points. Let $\mathcal{N}$ be an invertible sheaf on X with global sections $V = \mathrm{H}^0(X, \mathcal{N})$. We say $\mathcal{N}$ *separates a set of closed points* $d_1, d_2, \ldots, d_m \in X$ if, for $1 \leq i \leq m$, there exists $\alpha_i \in V$ such that $\alpha_i(d_j) = 0$ for $j \neq i$ but $\alpha_i(d_i) \neq 0$. By a slight abuse of notation we write this as $\alpha_i(d_j) = \delta_{ij}$ for $1 \leq i, j \leq m$.

Lemma 4.3. *Let $\mathcal{N}$ be a very ample invertible sheaf on X and set $1 + h = \dim_k \mathrm{H}^0(X, \mathcal{N})$. Let $\eta : X \hookrightarrow \mathbb{P}^h$ be the immersion corresponding to a basis of $\mathrm{H}^0(X, \mathcal{N})$. Then:*

(1) *$\mathcal{N}$ separates any pair of distinct points of X.*
(2) *$\mathcal{N}$ separates a triple a, b, c of distinct closed points of X if and only if the points $\eta(a), \eta(b), \eta(c)$ are not collinear in $\mathbb{P}^h$.*

Proof. (1) This is just [Ha, Proposition II.7.3].

(2) Each nonzero element $\alpha \in V = \mathrm{H}^0(\mathcal{N})$ corresponds to a hyperplane $Y \subseteq \mathbb{P}^h$ such that $\eta^{-1}(Y) = \{x \in X \mid \alpha(x) = 0\}$. Now $\eta(a), \eta(b), \eta(c)$ are collinear in $\mathbb{P}^h$ if and only if every hyperplane of $\mathbb{P}^h$ which contains $\eta(a)$ and $\eta(b)$ also contains $\eta(c)$, if and only if every $\alpha \in V$ such that $\alpha(a) = \alpha(b) = 0$ also satisfies $\alpha(c) = 0$. Since the ordering of a, b, c in this argument is immaterial this proves the result. $\square$

Lemma 4.4. *Assume that $\mathcal{L}$ is a σ-ample invertible sheaf on X. In order to prove that $\{\mathcal{R}_n\}$ is an ample sequence, it is enough to show that, for any $N \geq 0$, there is some $n_0 \geq 0$ such that the invertible sheaf $\mathcal{L}_n$ separates the points $c_{-N}, c_{-N+1}, \dots, c_{n-1}$ for all $n \geq n_0$.*

Proof. Recall from Section 2 that our convention is that $f^\sigma(x) = f(\sigma(x))$.

We first need to be careful about how the points separated by $\mathcal{L}$ are related to the points separated by $\mathcal{L}^{\sigma^n}$. So, suppose that an invertible sheaf $\mathcal{N}$ separates points $d_1, \dots, d_r$. For some $1 \leq i \leq d$, assume that $v \in \mathrm{H}^0(\mathcal{N})$ satisfies $v(d_j) = \delta_{ij}$ for $1 \leq j \leq d$. Then $w = v^{\sigma^m} \in \mathrm{H}^0(\mathcal{N})^{\sigma^m} \cong \mathrm{H}^0(\mathcal{N}^{\sigma^m})$ satisfies $w(\sigma^{-m}(d_j)) = v(d_j) = \delta_{ij}$ and so $\mathcal{N}^{\sigma^m}$ separates the points $\sigma^{-m}(d_j)$. In particular, if $\mathcal{N}$ separates $c_{-m}, \dots, c_r$, then $\mathcal{N}^{\sigma^m}$ separates $c_0, \dots, c_{m+r}$, since $c_i = \sigma^{-i}(c_0)$. Tensoring by a very ample sheaf $\mathcal{P}$ also preserves the property of separating a set of points. Indeed, if $v \in \mathrm{H}^0(\mathcal{N})$ is defined as above, pick any $u \in \mathrm{H}^0(\mathcal{P})$ such that $u(d_i) \neq 0$. Then $v \otimes u \in \mathrm{H}^0(\mathcal{N} \otimes \mathcal{P})$ certainly satisfies $(v \otimes u)(d_j) = \delta_{ij}$. Thus $\mathcal{N} \otimes \mathcal{P}$ separates the points $d_1, \dots, d_r$.

We now turn to the proof of the lemma. Fix an invertible sheaf $\mathcal{M}$ on X. Since $\mathcal{L}$ is σ-ample, there exists $m \geq 0$ such that $\mathcal{M} \otimes \mathcal{L}_m$ is very ample [Ke1, Proposition 2.3]. Taking $N = m$ in the hypothesis shows that $\mathcal{L}_{n-m}$ separates $c_{-m}, c_{-m+1}, \dots, c_{n-m-1}$ for $n \gg 0$. By the first paragraph, $(\mathcal{L}_{n-m})^{\sigma^m}$ separates $c_0, c_1, \dots, c_{n-1}$. Thus $(\mathcal{M} \otimes \mathcal{L}_m) \otimes (\mathcal{L}_{n-m})^{\sigma^m} = \mathcal{M} \otimes \mathcal{L}_n$ also separates $c_0, c_1, \dots, c_{n-1}$ for $n \gg 0$.

Write $\mathcal{N} = \mathcal{M} \otimes \mathcal{L}_n$ and consider the exact sequence

$$0 \to \mathcal{I}_n \otimes \mathcal{N} \to \mathcal{N} \to \mathcal{N}/\mathcal{I}_n\mathcal{N} \to 0. \tag{4.5}$$

Since $\mathcal{O}_X/\mathcal{I}_n \cong \bigoplus_{i=0}^{n-1} k(c_i)$ and $\mathcal{N}$ is invertible, there are isomorphisms

$$\mathcal{N}/\mathcal{I}_n\mathcal{N} \;\cong\; \bigoplus_{i=0}^{n-1} \mathcal{N}/\mathcal{I}^{\sigma^i}\mathcal{N} \;\cong\; \bigoplus_{i=0}^{n-1} k(c_i).$$

Thus $\alpha(c_j) = 0$ for some $\alpha \in \mathrm{H}^0(\mathcal{N})$ if and only if $\psi_j(\alpha) = 0$, where ψ_j is the map

$$\psi_j : \mathrm{H}^0(\mathcal{N}) \xrightarrow{\theta} \mathrm{H}^0(\mathcal{N}/\mathcal{I}_n\mathcal{N}) = \bigoplus_{i=0}^{n-1} \mathrm{H}^0(k(c_i)) \xrightarrow{\pi_j} \mathrm{H}^0(k(c_j)) \cong k.$$

The fact that $\mathcal{N}$ separates the set of points $\{c_0, c_1, \dots, c_{n-1}\}$ ensures that the map θ is surjective. But $\mathcal{L}$ is σ-ample and so $\mathrm{H}^1(\mathcal{N}) = 0$ for $n \gg 0$. From the long exact sequence

$$\mathrm{H}^0(\mathcal{N}) \xrightarrow{\theta} \mathrm{H}^0(\mathcal{N}/\mathcal{I}_n\mathcal{N}) \longrightarrow \mathrm{H}^1(\mathcal{I}_n \otimes \mathcal{N}) \longrightarrow \mathrm{H}^1(\mathcal{N}),$$

arising from (4.5), this implies that $\mathrm{H}^1(\mathcal{I}_n \otimes \mathcal{N}) = 0$ for $n \gg 0$. Thus $\mathrm{H}^1(\mathcal{M} \otimes \mathcal{R}_n) = 0$ for $n \gg 0$ and the result follows from Lemma 4.2. □

We can now prove in considerable generality that $\{\mathcal{R}_n\}$ is ample.

Proposition 4.6. *Assume that* (3.1) *holds. Let $\mathcal{L}$ be a very ample and σ-ample sheaf on X and assume that the Zariski closure of $\{c_i : i \geq 0\}$ is not isomorphic to $\mathbb{P}^1$. Then the sequence $\{\mathcal{R}_n\}$ is an ample sequence of bimodules.*

Proof. Since σ^m is an automorphism of X, the assumption on Zariski closures also ensures that the Zariski closure of $\{c_i : i \geq m\}$ is also not isomorphic to $\mathbb{P}^1$, for any $m \in \mathbb{Z}$. Consequently, for any closed immersion $\eta : X \hookrightarrow \mathbb{P}^t$, the Zariski closure of $\{\eta(c_i) : i \geq m\}$ in $\mathbb{P}^t$ cannot be contained in a line in $\mathbb{P}^t$. It is this consequence of our hypothesis that will actually be used in the proof. In particular, if $\mathcal{M}$ is any very ample invertible sheaf, then Lemma 4.3(2) implies that there exist three points from the set $\{c_i\}_{i \geq m}$ which are separated by $\mathcal{M}$.

The strategy of the proof is to apply Lemma 4.4; thus for $N \geq 0$ we need to show that, if n is sufficiently large, then for every $-N \leq k \leq n-1$ we can find $w \in \mathrm{H}^0(\mathcal{L}_n)$ such that $w(c_i) = \delta_{ik}$ for $-N \leq i \leq n-1$. The way we do this is to use the tensor product structure $\mathcal{L}_n = \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$ and, for each $0 \leq i < n$, find some $w_i \in \mathrm{H}^0(\mathcal{L}^{\sigma^i})$ such that $w_i(c_k) \neq 0$, but $w_i(c_j) = 0$ for "sufficiently many" $j \neq k$. Then $w = w_0 \otimes \cdots \otimes w_{n-1}$ will have the desired property.

Fix $N \geq 0$. Pick three distinct points d_{01}, d_{02}, d_{03} from the set $\{c_\ell : \ell \geq -N\}$ that are separated by $\mathcal{L}$. Then pick m_1 such that

$$\{d_{01}, d_{02}, d_{03}\} \subset \{c_{-N}, c_{-N+1}, \ldots, c_{m_1-1}\}.$$

Since $\mathcal{L}^\sigma$ is also very ample we may choose three distinct points d_{11}, d_{12}, d_{13} from $\{c_\ell : \ell \geq m_1\}$ that are separated by $\mathcal{L}^\sigma$. Continuing inductively, we may pick points $\{d_{ji} : 0 \leq j \leq N-1 \text{ and } 1 \leq i \leq 3\}$ such that the d_{ji} are all distinct and such that $\mathcal{L}^{\sigma^j}$ separates d_{j1}, d_{j2}, d_{j3} for each j. Note that this implies that, for any j and an arbitrary point $d \in X$, we can find $u, v \in \{1, 2, 3\}$ such that $\mathcal{L}^{\sigma^j}$ separates the points $\{d, d_{ju}, d_{jv}\}$.

Now choose n large enough so that all of the $\{d_{ji}\}$ are contained in the set $\{c_\ell : -N \leq \ell \leq n\}$ (where we have added an extra point for notational convenience). Fix $-N \leq k \leq n$. For each $0 \leq j \leq N-1$, the conclusion of the last paragraph implies that we can find two of the d_{ji}, say d_{j1} and d_{j2}, and some $w_j \in \mathrm{H}^0(\mathcal{L}^{\sigma^j})$ such that $w_j(c_k) \neq 0$ but $w_j(d_{j1}) = 0 = w_j(d_{j2})$. Now let $\{e_i\}$ denote some enumeration of the remaining $n - N$ points; more precisely write

$$\{c_\ell : -N \leq \ell \leq n\} = \{c_k\} \cup \{d_{j1}, d_{j2} : 0 \leq j \leq N-1\} \cup \{e_1, \ldots, e_{n-N}\}.$$

Since each $\mathcal{L}^{\sigma^j}$ is very ample, Lemma 4.3(1) implies that, for $N \leq j \leq n-1$, there exists $w_j \in \mathrm{H}^0(\mathcal{L}^{\sigma^j})$ such that $w_j(c_k) \neq 0$ but $w_j(e_{j-N+1}) = 0$.

Thus $w = w_0 \otimes w_1 \otimes \cdots \otimes w_{n-1}$ is an element of $\mathrm{H}^0(\mathcal{L} \otimes \mathcal{L}^\sigma \cdots \otimes \mathcal{L}^{\sigma^{n-1}}) = \mathrm{H}^0(\mathcal{L}_n)$ with the property that $w(c_i) = \delta_{ik}$ for $-N \leq i \leq n$. Since k is arbitrary this implies that $\mathcal{L}_n$ separates the set of points $c_{-N}, c_{-N+1}, \ldots, c_n$. Thus Lemma 4.4 can be applied to prove the ampleness of the sequence of bimodules $\{\mathcal{R}_n\}$. □

Proof of Theorem 4.1. We may assume that $\dim X \geq 1$. Suppose that $\{c_i : i \in \mathbb{Z}\}$ is critically dense. By Proposition 3.10 $\mathcal{R}$ is right noetherian and by Proposition 2.10 gr-$\mathcal{R}$ is an abelian category. Since $X \ncong \mathbb{P}^1$, the hypotheses of Proposition 4.6 are satisfied and that result implies that $\{\mathcal{R}_n\}$ is an ample sequence of bimodules. Thus all of the hypotheses of Theorem 2.12 are satisfied and so R is right noetherian with qgr-$\mathcal{R} \simeq$ qgr-R.

As was noted in (3.2), $\mathcal{R}^{\mathrm{op}} \cong \mathcal{R}(X, \sigma(c), \mathcal{L}^{\sigma^{-1}}, \sigma^{-1})$ and, by [Ke1, Corollary 5.1], $\mathcal{L}^{\sigma^{-1}}$ is σ^{-1}-ample. Thus the claims on the left follow from those on the right. This completes the proof of Theorem 4.1. □

Remark 4.7. Proposition 4.6 and hence Theorem 4.1 are not the best results possible. Using a similar but more complicated argument we can prove that $\{\mathcal{R}_n\}$ is an ample sequence as long as $\mathcal{L}$ is σ-ample, ample and generated by global sections, and the set $\{c_i\}$ is not all contained in a dimension 1 subscheme of X. We do not know what are the weakest possible assumptions on $\mathcal{L}$ and the $\{c_i\}$ under which the proposition will hold.

The results we have proved this far also give a partial converse to Theorem 4.1.

Proposition 4.8. *Keep the assumptions of* (3.1). *Suppose that $\mathcal{C} = \{c_i\}_{i \geq 0}$ is not critically dense, but that the Zariski closure of $\mathcal{C}$ is not isomorphic to $\mathbb{P}^1$. Let $\mathcal{L}$ be very ample and σ-ample. Then neither $\mathcal{R} = \mathcal{R}(X, c, \mathcal{L}, \sigma)$ nor $R = \Gamma(\mathcal{R})$ is right noetherian.*

Proof. By Theorem 3.10, $\mathcal{R}$ is not right noetherian. By Proposition 4.6 $\{\mathcal{R}_n\}$ is, however, a right-ample sequence of bimodules.

The final assertion of Theorem 3.10 provides an infinite proper ascending chain $\mathcal{M}^0 \subsetneq \mathcal{M}^1 \subsetneq \dots$ of coherent right $\mathcal{R}$-ideals such that the factors $\mathcal{M}^{n+1}/\mathcal{M}^n$ are not in tors-$\mathcal{R}$. Since $\{\mathcal{R}_n\}$ is an ample sequence, and these are coherent ideals, the proof of [VB1, Theorem 5.2, Step 1] shows that, for each n, $\mathcal{M}^n_i$ is generated by its global sections for $i \gg 0$. Writing $M^n = \Gamma(\mathcal{M}^n)$, this forces $M^n_i \subsetneq M^{n+1}_i$ for all $i \gg 0$, and so $M^0 \subsetneq M^1 \subsetneq \dots$ is also a proper ascending chain of right R-ideals. Thus R is not right noetherian. □

There is one minor case of Theorem 4.1 that will not be of interest in the sequel. This is when X is a curve or a point. The latter case is completely trivial, so assume that X is a curve. In this case, since X has an infinite automorphism, it is either rational or elliptic. In the former case it must also be singular. In either case, since c is a smooth point (Lemma 3.7), both $\mathcal{I} = \mathcal{I}_c$ and $\mathcal{N} = \mathcal{I} \otimes \mathcal{L}$ are invertible sheaves. Thus R is nothing more than the twisted homogeneous coordinate ring $B(X, \mathcal{N}, \sigma)$, as defined at the end of Section 2. Note that, as $\mathcal{L}$ is very ample, the fact that $X \ncong \mathbb{P}^1$ implies that $\mathcal{L}$ must have at least 3 global sections. An easy exercise then implies that $\mathcal{N}$ is ample. Hence Theorem 4.1 is just a very special case of [AV, Theorems 1.3 and 1.4] and $\mathcal{R}$ does not have any unusual properties. In contrast, the theorem does not hold for $X = \mathbb{P}^1$ since one can take $\mathcal{L} = \mathcal{I}^{-1}$.

The aim of the rest of the paper is to obtain a deeper understanding of the algebra $\mathcal{R}(X, c, \mathcal{L}, \sigma)$ and its section ring $R = \Gamma(\mathcal{R})$ under the assumptions of Theorem 4.1. By the last paragraph we are only interested in the case when $\dim X \geq 2$. Thus for the rest of the paper we will make the following assumptions:

Assumptions 4.9. Let X be a integral projective scheme of dimension $d \geq 2$. Fix $\sigma \in \mathrm{Aut}(X)$ and a very ample, σ-ample invertible sheaf $\mathcal{L}$. Finally assume that $c \in X$ is chosen so that $\mathcal{C} = \{c_i : i \in \mathbb{Z}\} = \{\sigma^{-i}(c) : i \in \mathbb{Z}\}$ is critically dense in X.

We will always write $\mathcal{R} = \mathcal{R}(X, c, \mathcal{L}, \sigma)$ and $R = \Gamma(\mathcal{R}) = R(X, c, \mathcal{L}, \sigma)$. By Theorem 4.1, R is noetherian with qgr-$R \simeq$ qgr-$\mathcal{R}$.

It is often useful to work with connected graded rings that are generated in degree one and we end the section by giving two ways in which this may be achieved; either by replacing R by a large Veronese ring or by assuming that the invertible sheaf $\mathcal{L}$ is "sufficiently ample."

Proposition 4.10. *Keep the hypotheses of* (4.9). *Then the Veronese ring* $R^{(p)} = \bigoplus_{i\geq 0} R_{pi}$ *is generated in degree* 1 *for some* $p \gg 0$.

Remark 4.11. The ring $R^{(p)}$ is not quite an algebra of the same form as R; explicitly $R^{(p)}$ is the global sections of the bimodule algebra $\bigoplus_{i\geq 0}((\mathcal{I}_p \otimes \mathcal{L}_p)_{\sigma^p})^{\otimes i}$ where $\mathcal{I}_p$ is the ideal sheaf defining the finite set of points $\{c_0, c_1, \ldots, c_{p-1}\}$.

Proof. Set $\mathcal{J}_t = \mathcal{I}_t \otimes \mathcal{L}_t$ for $t \geq 1$. By Proposition 4.6 we may chose $r \geq 1$ such that $\mathcal{J}_r$ is generated by its global sections. Thus, there exists a short exact sequence

$$0 \to \mathcal{V} \to \mathrm{H}^0(X, \mathcal{J}_r) \otimes \mathcal{O}_X \to \mathcal{J}_r \to 0,$$

of $\mathcal{O}_X$-modules, for some sheaf $\mathcal{V}$. Since $\{\mathcal{J}_n\}$ and hence $\{\mathcal{J}_n^{\sigma^r}\}$ is an ample sequence, there exists n_0 such that $\mathrm{H}^1(X, \mathcal{V} \otimes \mathcal{J}_n^{\sigma^r}) = 0$ for $n \geq n_0$. Hence, if one tensors the displayed exact sequence on the right with $\mathcal{J}_{nr}^{\sigma^r}$ for $nr \geq n_0$ and takes global sections, one obtains the exact sequence

$$\mathrm{H}^0(X, \mathcal{J}_r) \otimes \mathrm{H}^0(X, \mathcal{J}_{nr}^{\sigma^r}) \to \mathrm{H}^0(X, \mathcal{J}_{(n+1)r}) \to \mathrm{H}^1(X, \mathcal{V} \otimes \mathcal{J}_{nr}^{\sigma^r}).$$

By construction, this final term is zero, so the exact sequence is nothing more that the statement that the natural map $R_r \otimes R_{nr} \to R_{(n+1)r}$ is a surjection.

By induction, for all $j \geq 1$ and all $n \geq n_0$, we get $R_{jr}R_{nr} = R_{(n+j)r}$. This implies that $R^{(nr)}$ is generated in degree one; that is by R_{nr}. □

In the next result, we write $\mathcal{L}_n^m = (\mathcal{L}_n)^{\otimes m} \cong \mathcal{N}_n$ for $\mathcal{N} = \mathcal{L}^{\otimes m}$.

Proposition 4.12. *Keep the hypotheses of* (4.9). *Then there exists* $M \in \mathbb{N}$ *such that, for* $m \geq M$*:*

(1) $\mathcal{I}_n \otimes \mathcal{L}_n^m$ *is generated by its global sections for all* $n \geq 1$.
(2) $R(X, c, \mathcal{L}^m, \sigma)$ *is generated in degree* 1.

Proof. (1) By Proposition 4.6 there exists n_0 such that $\mathcal{I}_n \otimes \mathcal{L}_n$ is generated by its global sections for $n \geq n_0$. Since $\mathcal{L}_n$ is already globally generated, the sheaves $\mathcal{I}_n \otimes \mathcal{L}_n^m$ are globally generated for $n \geq n_0$ and $m \geq 1$. On the other hand, $\mathcal{L}_n$ is ample for all $n \geq 1$ and so there exists m_0 such that $\mathcal{I}_n \otimes \mathcal{L}_n^m$ is globally generated for all $1 \leq n \leq n_0$ and $m \geq m_0$. Combining these observations proves (1).

(2) The proof will use the following notion of Castelnuovo-Mumford regularity: An $\mathcal{O}_X$-module $\mathcal{F}$ is *r-regular with respect to a very ample invertible sheaf* $\mathcal{H}$ if $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{H}^{(r-i)}) = 0$ for $i > 0$. In this proof, all regularities will be taken with respect to $\mathcal{H} = \mathcal{L}^\sigma$. The minimum r such that $\mathcal{F}$ is r-regular is denoted $\operatorname{reg} \mathcal{F}$ and called the *regularity* of $\mathcal{F}$. Set $r = \max\{1, \operatorname{reg} \mathcal{O}_X\}$.

For any $m \geq m_0$, part 1 provides a short exact sequence

$$0 \to \mathcal{K}_m \to \mathrm{H}^0(\mathcal{I} \otimes \mathcal{L}^m) \otimes \mathcal{O}_X \to \mathcal{I} \otimes \mathcal{L}^m \to 0, \tag{4.13}$$

for some sheaf $\mathcal{K}_m$. Since $\mathcal{L}$ is very ample, there exists $m_1 \geq m_0$ such that $\mathcal{I} \otimes \mathcal{L}^m$ is 0-regular with respect to $\mathcal{L}^\sigma$ for all $m \geq m_1$. By [AK, Lemma 3.1] this implies that $\mathcal{K}_m$ is r-regular, independently of $m \geq m_1$.

We want to find similar upper bounds on the regularity of other sheaves. Since $\{(\mathcal{I}_n \otimes \mathcal{L}_n)^\sigma\}$ is an ample sequence, the regularity of $(\mathcal{I}_n \otimes \mathcal{L}_n)^\sigma$ is bounded above, independently of $n \geq 1$. Moreover, by the vanishing theorem of [Fj, Theorem 5.1], there is a universal upper bound on the regularity of *any* ample invertible sheaf. Thus, $\operatorname{reg}((\mathcal{L}_{n-1}^{m-1})^{\sigma^2})$ is bounded above, independently of $n \geq 1$ and $m \geq 2$. Finally, if $\mathcal{F}, \mathcal{G}$ are locally free except in a subscheme of dimension ≤ 2, then [Ke2, §2] shows that $\operatorname{reg}(\mathcal{F} \otimes \mathcal{G}) \leq \operatorname{reg} \mathcal{F} + \operatorname{reg} \mathcal{G} + t$, where $t = (r-1)(\dim X - 1)$. Note that each $\mathcal{I}_n$, and hence each $\mathcal{K}_m$, is locally free except in a subscheme of dimension 0. Combining these observations shows that $\operatorname{reg}(\mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n)^\sigma \otimes (\mathcal{L}_{n-1}^{m-1})^{\sigma^2})$ is bounded above, independently of $n \geq 1$ and $m \geq m_1$.

Thus there exists $M \geq m_1$ such that, for all $m \geq m_1$,

$$\begin{aligned} M &\geq \operatorname{reg}(\mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n)^\sigma \otimes (\mathcal{L}_{n-1}^{m-1})^{\sigma^2}) \\ &= \operatorname{reg}(\mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n)^\sigma \otimes (\mathcal{L}_{n-1}^{m-1})^{\sigma^2} \otimes (\mathcal{L}^\sigma)^{m-1}) + (m-1) \\ &= \operatorname{reg}(\mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma) + (m-1). \end{aligned}$$

In other words, $\operatorname{reg}(\mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma) \leq 1$ for all $m \geq M$. By [Kl, Proposition 1, p.307] this implies that $H^1(\mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma) = 0$.

Now tensor (4.13) with $(\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma$ and note that the resulting sequence is exact, by Lemma 3.3. Taking cohomology gives the exact sequence

$$\mathrm{H}^0(X, \mathcal{I} \otimes \mathcal{L}^m) \otimes \mathrm{H}^0(X, (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma) \xrightarrow{\theta} \mathrm{H}^0(X, \mathcal{I}_{n+1} \otimes \mathcal{L}_{n+1}^m) \longrightarrow$$
$$\longrightarrow H^1(\mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma).$$

By the conclusion of the last paragraph, this final term is zero and hence θ is surjective for $n \geq 1$ and $m \geq M$. This is equivalent to the assertion of part 2. $\square$

5. Naïve noncommutative blowing up

The hypotheses of Assumptions 4.9 will remain in force throughout this section. In the introduction we asserted that one should regard the bimodule algebra $\mathcal{R}$ as a sort of noncommutative blowup of X at the point c. In this section we justify and expand upon those comments, showing that they are easy consequences of the basic construction. One should note, however, that modules get slightly shifted and so it may be more natural to think of $\mathcal{R}$ as the blowup of c_{-1}, or perhaps better yet as the blowup of the entire orbit $\{c_n\}_{n \in \mathbb{Z}}$. We discuss this in more detail after Proposition 5.8.

As was noted in the introduction, one way to form the blowup $\widetilde{X}$ of X at the closed point $c \in X$ is to use the identity $\mathcal{O}_{\widetilde{X}}$-mod $=$ qgr-$\mathcal{A}$, where $\mathcal{A} = \bigoplus \mathcal{I}_c^n$. Since this bimodule algebra equals $\mathcal{R}(X, c, \mathcal{O}_X, \mathrm{id})$, it is natural to define qgr-$\mathcal{R}(X, c, \mathcal{O}_X, \sigma)$ to be *the naïve noncommutative blowup* of X at c. By Proposition 3.5, qgr-$\mathcal{R}(X, c, \mathcal{O}_X, \sigma) \simeq$ qgr-$\mathcal{R}(X, c, \mathcal{L}, \sigma)$ for any $\mathcal{L}$ and so Theorem 4.1 can be restated as:

Corollary 5.1. *Keep the hypotheses of* (4.9). *Then* qgr-$R(X, c, \mathcal{L}, \sigma)$ *is a naïve noncommutative blowup of X at c.* $\square$

Perhaps the biggest difference between the classical and naïve noncommutative blowups is in the properties of the inverse image of the smooth point that has been blown up. In the commutative case, of course, one gets a divisor. However, in our case we get just (a finite sum of copies of) one point, where we define a *point* in qgr-$\mathcal{R}$ to be a simple object in that category.

To explain this we need a further definition. As in (3.1), the skyscraper sheaf at a closed point $x \in X$ is written $k(x) = \mathcal{O}_X/\mathcal{I}_x$. Then one has a natural graded right $\mathcal{R}$-module

$$\overline{x} = k(x) \oplus k(x)_\sigma \oplus k(x)_{\sigma^2} \cdots . \tag{5.2}$$

The image of $\overline{x}$ in qgr-$\mathcal{R}$ will be written $\widetilde{x}$. It is an easy exercise to see that $\widetilde{x}$ is a simple object in qgr-$\mathcal{R}$ and we call it a *closed point* in qgr-$\mathcal{R}$. As will be seen in Theorem 6.7, all points in qgr-$\mathcal{R}$ are closed points, so the notation is reasonable.

Proposition 5.3. *Keep the hypotheses of* (4.9) *and let $x \in X$ be a closed point. Let $\widetilde{\rho} : \mathcal{O}_X$ -mod $\to$ qgr-$\mathcal{R}$ denote the blowup map defined by $\rho : \mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{R} \in$ gr-$\mathcal{R}$. Then:*

(1) *If $x \neq c_j$ for $j \geq 0$, then $\widetilde{\rho}(k(x))$ is the closed point $\widetilde{x}$ in* qgr-$\mathcal{R}$.
(2) *If $x = c_j$ for $j \geq 0$, then $\widetilde{\rho}(k(x))$ is a direct sum of $d = \dim X$ copies of the closed point $\widetilde{x}$ in* qgr-$\mathcal{R}$.

Remark 5.4. When $x = c_n$ for $n \in \mathbb{Z}$, we call $\widetilde{c}_n$ an *exceptional point.* In Theorem 6.7 we will slightly modify the functor $- \otimes \mathcal{R}$ in order to remove the direct sum that appears in part 2 of the proposition. This will show that the subcategory of torsion sheaves in $\mathcal{O}_X$ -Mod is equivalent to a corresponding subcategory of qgr-$\mathcal{R}$.

Proof. In the notation of Section 3, $\rho(k(x))$ has the following structure as a left $\mathcal{O}_X$-module:

$$\rho(k(x)) = \bigoplus (\mathcal{I}_n/\mathcal{I}_x\mathcal{I}_n) \otimes \mathcal{L}_n \cong \bigoplus (\mathcal{I}_n/\mathcal{I}_x\mathcal{I}_n)\,.$$

If $x \notin \{c_j : j \geq 0\}$, then $\mathcal{I}_x$ and $\mathcal{I}_n$ are comaximal for all $n \geq 0$ with $\mathcal{I}_n/\mathcal{I}_x\mathcal{I}_n \cong \mathcal{I}_n/\mathcal{I}_x \cap \mathcal{I}_n \cong \mathcal{O}_X/\mathcal{I}_x \cong k(x)$. Thus $\widetilde{\rho}(k(x)) = \widetilde{x}$.

On the other hand, if $x = c_j$ for some $j \geq 0$ then, for $n > j$, one has $\mathcal{I}_{c_j}\mathcal{I}_n = \mathcal{I}_{c_j}^2\mathcal{I}_n^*$, where $\mathcal{I}_n^* = \prod_{i=0}^{n-1}\{\mathcal{I}_{c_i} \mid i \neq j\}$. By Lemma 3.7, each c_i is a smooth point of X and so

$$\mathcal{I}_n/\mathcal{I}_{c_j}\mathcal{I}_n \cong \left(\mathcal{I}_{c_j} \cap \mathcal{I}_n^*\right) / \left(\mathcal{I}_{c_j}^2 \cap \mathcal{I}_n^*\right) \cong \mathcal{I}_{c_j}/\mathcal{I}_{c_j}^2 \cong \bigoplus_{r=1}^{d} k(c_j).$$

Thus $\widetilde{\rho}(k(x))$ is the direct sum of d copies of the point $\widetilde{c}_j$. $\square$

In the commutative case one way to see that the exceptional divisor really is a divisor is as follows: Let $\rho : \widetilde{X} \to X$ be the blowup of a smooth variety X at a point c and write $\widetilde{\mathcal{I}}_c = \rho^{-1}(\mathcal{I}_c) \cdot \mathcal{O}_{\widetilde{X}}$ for the inverse image ideal sheaf. Then [Ha, Proposition II.7.13] shows that, under the identification $\mathcal{O}_{\widetilde{X}}$ -mod = qgr-$\mathcal{A}$ for $\mathcal{A} = \bigoplus \mathcal{I}_c^n$, the sheaf $\widetilde{\mathcal{I}}_c$ is simply the twisting sheaf $\mathcal{O}_{\widetilde{X}}(1) = \bigoplus_{n\geq 1} \mathcal{I}_c^{n+1}$. As such it is invertible and so corresponds to a (Cartier) divisor.

A similar argument works in our situation with the exception of the last sentence: there is no correspondence between invertible objects and divisors. To explain this it is enough to work with $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{S} = \mathcal{R}(X, c, \mathcal{O}_X, \sigma)$ and so for most of this section we will work with that bimodule algebra. As we will see, the exceptional

point $\widetilde{c}_{-1}$ of Proposition 5.3 is indeed equal to $\mathcal{S}/\mathcal{K}$ where $\mathcal{K}$ is naturally isomorphic to the shift $\mathcal{S}[1]$. Similar results hold for each $\widetilde{c}_n$.

It is natural to write the shifts $\mathcal{S}[m]$ in the form $\bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^n$, for some $\mathcal{O}_X$-modules $\mathcal{F}_n$ but, as the next lemma shows, one has to be careful about the powers of σ appearing in the $\mathcal{F}_n$.

Lemma 5.5. *Suppose that $\mathcal{N}$ is a right module over $\mathcal{R} = \mathcal{R}(X, c, \mathcal{L}, \sigma)$, for $\mathcal{L}$ arbitrary, that can be written in the form $\mathcal{N} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ for some $\mathcal{O}_X$-modules $\mathcal{F}_n$ with the trivial bimodule structure. If $m \in \mathbb{Z}$, then $\mathcal{N}[m] \cong \bigoplus \mathcal{G}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ where:*
$\mathcal{G}_n = (\mathcal{F}_{n+m} \otimes \mathcal{L}_m)^{\sigma^{-m}}$ (with the trivial bimodule structure) if $m > 0$ and
$\mathcal{G}_n = (\mathcal{F}_{n+m})^{\sigma^{-m}} \otimes \mathcal{L}_{-m}^{-1}$ if $m < 0$.

Proof. The result is well-known for the bimodule algebra $\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$, as defined at the end of Section 2. In particular, when $m \geq 0$ the result for $\mathcal{B}$ follows immediately from [SV, (3.1)] and the same argument works for $\mathcal{R}$. A simple computation then gives the required formula for $m \leq 0$. $\square$

Using this lemma we find that, at least in qgr-$\mathcal{S}$, the shift $\mathcal{S}[n]$ is isomorphic to the following right $\mathcal{S}$-module $\widetilde{\mathcal{K}}(n)$.

Definition 5.6. Let $\mathcal{S} = \mathcal{R}(X, c, \mathcal{O}, \sigma) \cong \bigoplus_{r \geq 0} \mathcal{I}\mathcal{I}^\sigma \cdots \mathcal{I}^{\sigma^{r-1}} \otimes \mathcal{O}_\sigma^{\otimes r}$ for $\mathcal{O} = \mathcal{O}_X$. For $n \in \mathbb{Z}$, define $\mathcal{O}_X$-bimodules

$$\widetilde{\mathcal{K}}(n) = \bigoplus_{r \geq 1+|n|} \mathcal{I}^{\sigma^{-n}} \mathcal{I}^{\sigma^{-n+1}} \cdots \mathcal{I}^{\sigma^{r-1}} \otimes \mathcal{O}_\sigma^{\otimes r} \cong \bigoplus_{r \geq 1+|n|} \left(\mathcal{I}^{\sigma^{-n}} \cdots \mathcal{I}^{\sigma^{r-1}} \right)_{\sigma^r}$$

and

$$\widetilde{\mathcal{K}}(n)^* = \bigoplus_{r \geq 1+|n|} \mathcal{I}\mathcal{I}^\sigma \cdots \mathcal{I}^{\sigma^{r-n-1}} \otimes \mathcal{O}_\sigma^{\otimes r} \cong \bigoplus_{r \geq 1+|n|} \left(\mathcal{I} \cdots \mathcal{I}^{\sigma^{r-n-1}} \right)_{\sigma^r}$$

The next result describes the bimodule structure of $\widetilde{\mathcal{K}}(n)$ and requires the following definitions. A module $\mathcal{M} \in$ qgr-$\mathcal{B}$ is called *an invertible $(\mathcal{A}, \mathcal{B})$-bimodule in qgr* if it is the image under the quotient functor $\pi_\mathcal{B}$ of an $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ and there exists a $(\mathcal{B}, \mathcal{A})$-bimodule $\mathcal{M}'$ such that, up to torsion, $\mathcal{M} \otimes_\mathcal{B} \mathcal{M}' \cong \mathcal{A}$ and $\mathcal{M}' \otimes_\mathcal{A} \mathcal{M} \cong \mathcal{B}$. The module $\mathcal{M}'$ is called the *inverse* of $\mathcal{M}$.

Proposition 5.7. *Set $\mathcal{S} = \mathcal{R}(X, c, \mathcal{O}, \sigma)$ and $\mathcal{S}' = \mathcal{R}(X, c_{-n}, \mathcal{O}, \sigma)$. Then $\widetilde{\mathcal{K}}(n)$ is an invertible $(\mathcal{S}', \mathcal{S})$-bimodule in qgr with inverse $\widetilde{\mathcal{K}}(n)^*$.*

Proof. By Lemma 3.4, $\mathcal{S}_1 \otimes \widetilde{\mathcal{K}}(n)^*_r = \mathcal{I} \otimes {}_1\mathcal{O}_\sigma \otimes \widetilde{\mathcal{K}}(n)^*_r = \widetilde{\mathcal{K}}(n)^*_{r+1}$, for any r such that $\widetilde{\mathcal{K}}(n)^*_r \neq 0$. Thus $\widetilde{\mathcal{K}}(n)^*$ is a left $\mathcal{S}$-module and, similarly, $\widetilde{\mathcal{K}}(n)$ is a right $\mathcal{S}$-module. If $\mathcal{H} = \mathcal{I}^{\sigma^{-n}}$, then $\mathcal{S}'$ is the bimodule algebra $\bigoplus_{r \geq 0} \mathcal{H}\mathcal{H}^\sigma \cdots \mathcal{H}^{\sigma^{r-1}} \otimes \mathcal{O}_\sigma^{\otimes r}$. From its definition one finds that

$$\widetilde{\mathcal{K}}(n) = \bigoplus_{r \geq 1+|n|} \mathcal{H}\mathcal{H}^\sigma \cdots \mathcal{H}^{\sigma^{n+r-1}} \otimes \mathcal{O}_\sigma^{\otimes r},$$

which is obviously a left $\mathcal{S}'$-module. A similar argument shows that $\widetilde{\mathcal{K}}(n)^*$ is a right $\mathcal{S}'$-module and the left and right actions are clearly compatible.

Now consider $\widetilde{\mathcal{K}}(n)^* \otimes_{\mathcal{S}'} \widetilde{\mathcal{K}}(n)$. For t sufficiently large, Lemma 2.3 implies that its t^{th} summand is

$$\sum_i \left(\mathcal{I} \cdots \mathcal{I}^{\sigma^{i-n-1}}\right)_{\sigma^i} \left(\mathcal{I}^{\sigma^{-n}} \cdots \mathcal{I}^{\sigma^{t-i-1}}\right)_{\sigma^{t-i}} = \sum_i \left(\mathcal{I} \cdots \mathcal{I}^{\sigma^{t-1}}\right)_{\sigma^t} = \mathcal{S}_t,$$

as required. The proof that $\widetilde{\mathcal{K}}(n) \otimes_{\mathcal{S}} \widetilde{\mathcal{K}}(n)^* \cong \mathcal{S}'$ in qgr-$\mathcal{S}'$ is essentially the same. $\square$

The point in constructing the $\widetilde{\mathcal{K}}(n)$ was to show that the exceptional point $\widetilde{c}_{-1}$ could be written as $\mathcal{S}/\widetilde{\mathcal{K}}(1)$. In fact we have a more general result:

Proposition 5.8. *In* qgr-$\mathcal{S}$ *one has* $\widetilde{c}_{-n-1} \cong \widetilde{\mathcal{K}}(n)/\widetilde{\mathcal{K}}(n+1)$ *for all* $n \in \mathbb{Z}$.

Proof. Since the $\mathcal{I}^{\sigma^i}$ are comaximal, $\mathcal{I}^{\sigma^{-n}} \cdots \mathcal{I}^{\sigma^{-1}} / \mathcal{I}^{\sigma^{-n-1}} \cdots \mathcal{I}^{\sigma^{-1}} \cong \mathcal{O}_X/\mathcal{I}^{\sigma^{-n-1}}$, for all n. Thus in qgr-$\mathcal{S}$ we have

$$\widetilde{\mathcal{K}}(n)/\widetilde{\mathcal{K}}(n+1) \;\cong\; \bigoplus_{r \geq 2+|n|} \left(\mathcal{O}_X/\mathcal{I}^{\sigma^{-n-1}}\right)_{\sigma^r} \;=\; \widetilde{c}_{-n-1}. \qquad \square$$

We remarked in the introduction to this section that there is some ambiguity in what is actually been blown up in the passage from mod-$\mathcal{O}_X$ to qgr-$\mathcal{S}$. Given the way the construction works, we feel the correct interpretation is that we have just naïvely blown up the point c but, in the process, we automatically blow up the full orbit $\{c_i : i \in \mathbb{Z}\}$. This is illustrated by Proposition 5.8: all the modules $\widetilde{c}_n \in$ qgr-$\mathcal{S}$ can be written as a factor of two invertible bimodules and so, in this respect, they are more similar to a divisor than to a point. Another way of viewing the same result (which can also be proved directly from Lemma 5.5) is that $\widetilde{c}_n = \widetilde{c}[-n]$.

Remark 5.9. In the discussion above we have concentrated on modules over $\mathcal{S}$ since this most naturally corresponds to the commutative description of blowing up. In fact, one can also define the commutative blow-up in terms of $\bigoplus \mathcal{I}^n \mathcal{L}^{\otimes n}$ (see [Ha, Lemma II.7.9]) and so one should expect that these results for $\mathcal{S}$ have natural analogues for $\mathcal{R} = \mathcal{R}(X, c, \mathcal{L}, \sigma)$. This is true. We leave it to the reader to check that Definition 5.6–Proposition 5.8 all remain true as results about $\mathcal{R}$-modules if one simply replaces $\mathcal{O}$ by $\mathcal{L}$ and $\mathcal{S}$ by $\mathcal{R}$ in their statements and adjusts the proofs accordingly. Note that the new module $\widetilde{\mathcal{K}}(n)$ will not be isomorphic to $\mathcal{R}[n]$ in qgr-$\mathcal{R}$.

There is another version of noncommutative blowing up, due to Van den Bergh [VB2], that does have the properties expected of blowing up and his theory has been very useful in describing noncommutative surfaces. While Van den Bergh's construction is notationally very similar to ours, it is actually rather different. For one thing, it only works in the following situation: one has a connected graded ring S of Gelfand-Kirillov dimension 3 that surjects onto a twisted homogeneous coordinate ring $B = B(E, \mathcal{N}, \tau)$, where E is a curve, and the aim is to blow up a point c on that curve. This requires the construction of something analogous to the graded algebra $\bigoplus \mathcal{I}_c^n$. However, since B is noncommutative, $\bigoplus \mathcal{I}_c^n$ does not carry a natural algebra structure and so one has to be more subtle. Rather than use a category like qgr-S, Van den Bergh works in the category of left exact functors from qgr-S to itself and so, in particular, $\pi(S)$ is replaced by the identity functor on qgr-S. It is then nontrivial to show that the blowup has the expected properties. In particular, the inverse image of c does look like a divisor. The details can be

found in [VB2] and a brief introduction to the construction and its applications are described in [SV, Section 13].

6. $\mathcal{R}$-modules and equivalences of categories

The hypotheses from Assumptions 4.9 will remain in force throughout this section. One nice consequence of critical density is that it forces modules over $\mathcal{R} = \mathcal{R}(X, c, \mathcal{L}, \sigma)$ and $R = \Gamma(\mathcal{R})$ to have a very pleasant structure; indeed in many cases they are just induced from $\mathcal{O}_X$-modules. This will be used in this section to give various equivalences of categories, notably that the category of coherent torsion $\mathcal{O}_X$-modules is equivalent to the subcategory of Goldie torsion modules in qgr-$\mathcal{R}$, as defined below. This gives the promised improvement of Proposition 5.3. We also give a natural analogue of the standard fact that, for a blowup $\rho : \widetilde{X} \to X$ at a smooth point x, the schemes $X \smallsetminus \{x\}$ and $\widetilde{X} \smallsetminus \rho^{-1}(x)$ are isomorphic. For these results it would be sufficient, by Proposition 3.5, to work with just $\mathcal{R}(X, c, \mathcal{O}_X, \sigma)$, but we will work in the general case since this will enable us to draw conclusions about R-modules.

If A is a noetherian graded domain, a graded A-module M is called *Goldie torsion* (to distinguish this from the notion of torsion already defined) if every homogeneous element of M is killed by some nonzero homogeneous element of A. Equivalently, M is a direct limit of modules of the form $(A/I)[n]$, for nonzero graded right ideals I. The latter notion passes to all the categories Q we consider; for example a right $\mathcal{R}$-module is Goldie torsion if it is a direct limit of modules of the form $(\mathcal{R}/\mathcal{K})[n]$ for nonzero right ideals $\mathcal{K}$ of $\mathcal{R}$. Of course, Goldie torsion $\mathcal{O}_X$-modules are just the torsion $\mathcal{O}_X$-modules, as in [Ha, Exercise II.6.12]. We write $\mathrm{GT}\, Q$ for the full subcategory of Goldie torsion modules in Q.

We start by giving some technical results on the structure of Goldie torsion modules. If $\mathcal{N} = \bigoplus \mathcal{N}_n \in \text{Gr-}\mathcal{R}$, recall from (2.6) that we may write each $\mathcal{N}_n$ as an $\mathcal{O}_X$-bimodule of the form $(\mathcal{G}_n)_{\sigma^n}$. It is often convenient to write $(\mathcal{G}_n)_{\sigma^n} = \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n}$, where $\mathcal{F}_n = {}_1(\mathcal{F}_n)_1$ has trivial bimodule structure and, as usual, $\mathcal{L}_\sigma^{\otimes n} = ({}_1\mathcal{L}_\sigma)^{\otimes n}$. This has the advantage that the module structure of $\mathcal{N}$ is now given by maps of (left) sheaves $\mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \to \mathcal{F}_{n+1}$ for all n.

Lemma 6.1. (1) *If $\mathcal{N} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in \mathrm{GT}\,\text{gr-}\mathcal{R}$, then there exists a single module $\mathcal{F} \in \mathrm{GT}\,\mathcal{O}_X\text{-mod}$ such that $\mathcal{F}_n = \mathcal{F}$ for all $n \gg 0$.*

(2) *Conversely, if $\mathcal{F} \in \mathrm{GT}\,\mathcal{O}_X\text{-mod}$, then $\bigoplus_{n=0}^\infty \mathcal{F} \otimes \mathcal{L}_\sigma^{\otimes n} \in \mathrm{GT}\,\text{gr-}\mathcal{R}$.*

Proof. (1) Clearly $\mathcal{F}_n$ is Goldie torsion for $n \gg 0$. Since $\mathcal{N}$ is coherent, by Lemma 3.9 there is a surjection $\mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \twoheadrightarrow \mathcal{F}_{n+1}$ for $n \gg 0$ and so the supports satisfy $\operatorname{supp} \mathcal{F}_{n+1} \subseteq \operatorname{supp} \mathcal{F}_n$ for all $n \gg 0$. Since $\mathcal{C} = \{c_i\}_{i \geq 0}$ is critically dense, $\mathcal{C} \cap \operatorname{supp} \mathcal{F}_n$ is finite for each n such that $\mathcal{F}_n$ is torsion. Thus, $c_n \notin \operatorname{supp} \mathcal{F}_n$ for $n \gg 0$. By Lemma 3.3(2) and the fact that $\mathcal{F}_n / \mathcal{F}_n \mathcal{I}^{\sigma^n}$ is supported on $(\operatorname{supp} \mathcal{F}_n) \cap \{c_n\} = \emptyset$, one has $\mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \cong \mathcal{F}_n \mathcal{I}^{\sigma^n} = \mathcal{F}_n$ for all $n \gg 0$. Thus, we obtain a surjection $\mathcal{F}_n \twoheadrightarrow \mathcal{F}_{n+1}$ for all $n \geq n_0$. Since the $\mathcal{F}_n$ are noetherian this forces $\mathcal{F}_n \cong \mathcal{F}_{n+1}$ for $n \gg n_0$.

(2) Since $Y = \operatorname{supp} \mathcal{F}$ is a proper closed subset of X and the points $\{c_i\}_{i \geq 0}$ are critically dense, $Y \cap \{c_i\}_{i \geq n} = \emptyset$ for $n \gg 0$. So, just as in part 1, $\mathcal{F} \otimes \mathcal{I}^{\sigma^n} \cong \mathcal{F} \mathcal{I}^{\sigma^n} \cong \mathcal{F}$ for $n \gg 0$. Thus, if $\mathcal{M} = \bigoplus_{n=0}^\infty \mathcal{F} \otimes \mathcal{L}_\sigma^{\otimes n}$, then there is a surjection $\mathcal{M}_n \otimes \mathcal{R} \twoheadrightarrow \mathcal{M}_{\geq n}$ for $n \gg 0$ and $\mathcal{M}$ is coherent by Lemma 3.9. By construction, $\mathcal{M}$ is a Goldie torsion $\mathcal{R}$-module. $\square$

Lemma 6.2. *If $\mathcal{M} \in \text{gr-}\mathcal{R}$, then $\mathcal{M}$ has a finite filtration of submodules $0 = \mathcal{M}^0 \subset \mathcal{M}^1 \subset \cdots \subset \mathcal{M}^r = \mathcal{M}$ such that the factors $\mathcal{M}^i/\mathcal{M}^{i-1}$ are equal to either a shift $\mathcal{R}[j]$ of $\mathcal{R}$ or to a Goldie torsion module.*

Proof. This is similar to the analogous result for finitely generated modules over rings and the proof is left to the reader. □

The point of the next lemma is that, at least in large degree, the structure of an $\mathcal{R}$-module may be written using products instead of tensor products.

Lemma 6.3. *Let $\mathcal{N} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in \text{gr-}\mathcal{R}$. Then, under the natural map,*

$$\mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \cong \mathcal{F}_n\mathcal{I}^{\sigma^n} = \mathcal{F}_{n+1} \qquad \textit{for} \quad n \gg 0.$$

Proof. Let $0 \to \mathcal{H}' \to \mathcal{H} \to \mathcal{H}'' \to 0$ be an exact sequence of sheaves on X, and consider the commutative diagram

$$\begin{tikzcd}
\mathcal{H}' \otimes \mathcal{I}^{\sigma^n} \arrow[r] \arrow[d, "\theta_1"] & \mathcal{H} \otimes \mathcal{I}^{\sigma^n} \arrow[r] \arrow[d, "\theta_2"] & \mathcal{H}'' \otimes \mathcal{I}^{\sigma^n} \arrow[r] \arrow[d, "\theta_3"] & 0 \\
\mathcal{H}'\mathcal{I}^{\sigma^n} \arrow[r, "\phi"] & \mathcal{H}\mathcal{I}^{\sigma^n} \arrow[r] & \mathcal{H}''\mathcal{I}^{\sigma^n} &
\end{tikzcd}$$

where the θ_i are the natural surjections. Although the bottom row is not in general exact, the map ϕ is injective. A diagram chase then shows that, if θ_1 and θ_3 are isomorphisms, then θ_2 is an isomorphism.

If $\mathcal{N}$ is a Goldie torsion module, then the proof of Lemma 6.1 shows that $\mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \cong \mathcal{F}_n\mathcal{I}^{\sigma^n}$ for $n \gg 0$. Alternatively, suppose that $\mathcal{N} = \mathcal{R}[m]$ is a shift of $\mathcal{R}$. Lemma 5.5 implies that $\mathcal{F}_n = \mathcal{I}^{\sigma^{-m}} \cdots \mathcal{I}^{\sigma^{n-1}} \otimes \mathcal{L}^{\alpha}_{|m|}$ for $n > |m|$ and the appropriate α. Lemma 3.3(2) therefore implies that the map $\mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \to \mathcal{F}_n\mathcal{I}^{\sigma^n}$ must be an isomorphism for $n > |m|$.

If $\mathcal{N}$ is an arbitrary coherent $\mathcal{R}$-module, then by Lemma 6.2 it has a finite filtration by shifts of $\mathcal{R}$ and Goldie torsion modules. It therefore follows by induction from the last two paragraphs that $\mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \to \mathcal{F}_n\mathcal{I}^{\sigma^n}$ is an isomorphism for $n \gg 0$. Since $\mathcal{N}$ is coherent, the induced maps $\phi_n : \mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} = \mathcal{F}_n\mathcal{I}^{\sigma^n} \to \mathcal{F}_{n+1}$ defining the module structure of $\mathcal{N}$ are surjections for all $n \geq n_0$.

It remains to prove that the surjections $\phi_n : \mathcal{F}_n\mathcal{I}^{\sigma^n} \to \mathcal{F}_{n+1}$ are isomorphisms for $n \gg n_0$. Pulling back to $\mathcal{F}_{n_0}$, we may write $\mathcal{F}_n = \mathcal{A}_n/\mathcal{B}_n$, for submodules $\mathcal{B}_n \subseteq \mathcal{A}_n \subseteq \mathcal{F}_{n_0}$. Since $\mathcal{F}_{n+1}$ is a homomorphic image of $\mathcal{F}_n\mathcal{I}^{\sigma^n} = (\mathcal{A}_n\mathcal{I}^{\sigma^n} + \mathcal{B}_n)/\mathcal{B}_n$, we find that $\mathcal{B}_{n+1} \supseteq \mathcal{B}_n$ for each $n \geq n_0$. Since $\mathcal{F}_{n_0}$ is noetherian, $\mathcal{B}_n = \mathcal{B}_{n+1}$ for all $n \gg n_0$, and hence $\mathcal{F}_n\mathcal{I}^{\sigma^n} \cong \mathcal{F}_{n+1}$ for all such n. □

As might be expected, the fact that $\mathcal{R}$-modules have a nice form is also reflected in the homomorphism groups. The next lemma collects the relevant facts. Recall from (2.7) that the natural map from Gr-$\mathcal{R}$ to Qgr-$\mathcal{R}$ is denoted π.

Lemma 6.4. *Let $\mathcal{N} = \bigoplus \mathcal{G}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in \text{Gr-}\mathcal{R}$. Then:*

(1) *There is a natural isomorphism*

$$\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{N})) \cong \lim_{n\to\infty} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{G}_n),$$

where the limit is induced from the multiplication map; specifically it sends $\theta \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{G}_n)$ to the map

$$\theta \otimes \mathcal{I}^{\sigma^n} : \mathcal{I}_{n+1} \cong \mathcal{I}_n \otimes \mathcal{I}^{\sigma^n} \to \mathcal{G}_n \otimes \mathcal{I}^{\sigma^n} \to \mathcal{G}_{n+1}.$$

(2) *Suppose that* $\mathcal{M} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ *and* $\mathcal{N}$ *are coherent Goldie torsion and, by Lemma 6.1, write* $\mathcal{F}_n = \mathcal{F}$ *and* $\mathcal{G}_n = \mathcal{G}$ *for* $n \geq n_0$. *Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Qgr}\text{-}\mathcal{R}}(\pi(\mathcal{M}), \pi(\mathcal{N})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Proof. The definition of homomorphisms in quotient categories implies that

$$\mathrm{Hom}_{\mathrm{Qgr}\text{-}\mathcal{R}}(\pi(\mathcal{M}), \pi(\mathcal{N})) = \lim_{n\to\infty} \mathrm{Hom}_{\mathrm{Gr}\text{-}\mathcal{R}}(\mathcal{M}_{\geq n}, \mathcal{N}), \tag{6.5}$$

whenever $\mathcal{M}$ is coherent (see, for example, [VB2, p.31]). On the other hand, we claim that there are natural vector space maps

$$\mathrm{Hom}_{\mathrm{Gr}\text{-}\mathcal{R}}(\mathcal{M}_{\geq n}, \mathcal{N}) \xrightarrow{\phi_n} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}_n, \mathcal{N}_n) \xrightarrow{\rho_n} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{G}_n). \tag{6.6}$$

Indeed, if $f \in \mathrm{Hom}_{\mathrm{Gr}\text{-}\mathcal{R}}(\mathcal{M}_{\geq n}, \mathcal{N})$, then f is a morphism of right $\mathcal{O}_X$-modules, so we may define $\phi_n(f)$ to be the restriction of f to $\mathcal{M}_n$. The map ρ_n is the natural isomorphism obtained by tensoring with the invertible bimodule $(\mathcal{L}_\sigma^{\otimes n})^{-1}$.

(1) The result will follow from (6.5) and (6.6) once we prove that the map $\rho_n \circ \phi_n$ is an isomorphism for $n \gg 0$. In this case, $\mathcal{F}_n = \mathcal{I}_n$ and, by Lemma 3.4, $\mathcal{I}_{n+r} \cong \mathcal{I}_n \otimes \mathcal{I}_r^{\sigma^n}$, for any $n, r \geq 0$. Thus, if $g \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{G}_n)$, then g induces a unique map

$$\mathcal{R}_{n+r} \cong \mathcal{I}_n \otimes (\mathcal{I}_r^{\sigma^n} \otimes \mathcal{L}_\sigma^{\otimes(n+r)}) \to \mathcal{G}_n \otimes (\mathcal{I}_r^{\sigma^n} \otimes \mathcal{L}_\sigma^{\otimes(n+r)}) \to \mathcal{G}_{n+r} \otimes \mathcal{L}_\sigma^{\otimes(n+r)} = \mathcal{N}_{n+r},$$

for any $r \geq 0$. This clearly defines an $\mathcal{R}$-module map $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{R}_{\geq n}, \mathcal{N})$ such that $\rho_n\phi_n(f) = g$. Thus $\rho_n\phi_n$ is in fact an isomorphism for all $n \geq 0$.

(2) The proof is essentially the same as that of part 1. Any element $g \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ determines a unique map $\mathcal{F} \otimes \mathcal{L}_\sigma^n \to \mathcal{G} \otimes \mathcal{L}_\sigma^n$ for $n \geq n_0$. For such an n this defines an $\mathcal{R}$-module map $f : \mathcal{M}_{\geq n} \to \mathcal{N}_{\geq n}$ with $\rho_n\phi_n(f) = g$. Thus $\rho_n\phi_n$ is an isomorphism for $n \geq n_0$ and the result follows from (6.5) and (6.6). $\square$

It is now easy to define an equivalence of categories between $\mathrm{GT}\,\mathrm{qgr}\text{-}R$ and $\mathrm{GT}\,\mathcal{O}_X\text{-mod}$.

Theorem 6.7. *Keep the hypotheses from* (4.9). *Then there are equivalences of categories*

$$\mathrm{GT}\,\mathrm{Qgr}\text{-}R \;\simeq\; \mathrm{GT}\,\mathrm{Qgr}\text{-}\mathcal{R} \;\simeq\; \mathrm{GT}\,\mathcal{O}_X\text{-Mod},$$

which restrict to equivalences $\mathrm{GT}\,\mathrm{qgr}\text{-}R \simeq \mathrm{GT}\,\mathrm{qgr}\text{-}\mathcal{R} \simeq \mathrm{GT}\,\mathcal{O}_X\text{-mod}$. *This equivalence is given by mapping* $\mathcal{F} \in \mathrm{GT}\,\mathcal{O}_X\text{-Mod}$ *to* $\pi\,(\bigoplus \mathcal{F} \otimes \mathcal{L}_\sigma^{\otimes n}) \in \mathrm{Qgr}\text{-}\mathcal{R}$.

Remark 6.8. For any closed point $x \in X$, this equivalence sends $k(x) \in \mathcal{O}_X\text{-mod}$ to $\widetilde{x} \in \mathrm{GT}\,\mathrm{qgr}\text{-}\mathcal{R}$ and so it does give the promised refinement of Proposition 5.3. Since the simple objects in $\mathcal{O}_X\text{-Mod}$ are precisely these modules $k(x)$, this also proves part 4 of Theorem 1.1.

Proof. The equivalence of categories Theorem 4.1 clearly restricts to an equivalence $\mathrm{GT}\,\mathrm{Qgr}\text{-}R \simeq \mathrm{GT}\,\mathrm{Qgr}\text{-}\mathcal{R}$, so only the second equivalence needs proving.

Define $\theta : \mathrm{GT}\,\mathcal{O}_X\text{-mod} \to \mathrm{GT}\,\mathrm{qgr}\text{-}\mathcal{R}$ by $\mathcal{F} \mapsto \pi(\bigoplus \mathcal{F} \otimes \mathcal{L}_\sigma^{\otimes n})$. That θ lands in $\mathrm{GT}\,\mathrm{qgr}\text{-}\mathcal{R}$ rather than $\mathrm{GT}\,\mathrm{Qgr}\text{-}\mathcal{R}$ follows from Lemma 6.1(2), and θ is clearly functorial since $- \otimes (\bigoplus \mathcal{L}_\sigma^{\otimes n})$ and π are functors. Lemma 6.1(1) shows that θ is surjective on objects. Finally, it follows from Lemma 6.4(2) that θ is full and faithful on morphisms, so that θ is an equivalence.

By [SV, Theorem 1.1.1], $\mathrm{GT}\,\mathrm{Qgr}\text{-}\mathcal{R}$ is the closure of $\mathrm{GT}\,\mathrm{qgr}\text{-}\mathcal{R}$ under direct limits and similarly for $\mathrm{GT}\,\mathcal{O}_X\text{-Mod}$. Thus, the equivalence $\mathrm{GT}\,\mathcal{O}_X\text{-mod} \simeq \mathrm{GT}\,\mathrm{qgr}\text{-}\mathcal{R}$

extends to an equivalence $\mathrm{GT}\,\mathcal{O}_X\text{-Mod} \simeq \mathrm{GT}\,\mathrm{Qgr}\text{-}\mathcal{R}$. Since direct limits commute with tensor products, the equivalence does still have the specified form. $\square$

A standard fact in geometry is that if $\rho : \widetilde{X} \to X$ is the blowup of X at a smooth point x, then $X \smallsetminus \{x\}$ is isomorphic to $\widetilde{X} \smallsetminus \rho^{-1}(x)$ [Ha, Proposition II.7.13]. The final result of this section proves the analogous result for qgr-$\mathcal{R}$. As may be expected from the results of Section 5 we have to remove all of the c_i from X rather than just one point. Thus we define C_X to be the smallest localizing subcategory of $\mathcal{O}_X$-Mod containing all of the $\{k(c_i)\}_{i\in\mathbb{Z}}$. The reader may check that the objects in this subcategory are exactly those quasicoherent sheaves which are supported at the set of points $\{c_i\}_{i\in\mathbb{Z}}$ and so C_X is a subcategory of $\mathrm{GT}\,\mathcal{O}_X$-Mod.

Similarly, write $C_{\mathcal{R}}$ for the localizing subcategory of Qgr-$\mathcal{R}$ generated by the modules $\widetilde{c}_n$ for $n \in \mathbb{Z}$. By Remark 6.8, the equivalence $\mathrm{GT}\,\mathcal{O}_X\text{-Mod} \simeq \mathrm{GT}\,\mathrm{Qgr}\text{-}\mathcal{R}$ restricts to an equivalence $C_X \simeq C_{\mathcal{R}}$.

Proposition 6.9. *Keep the hypotheses from* (4.9)*. There is an equivalence of categories* $\mathcal{O}_X\text{-Mod}/C_X \simeq \mathrm{Qgr}\text{-}\mathcal{R}/C_{\mathcal{R}}$.

Proof. Given a sheaf $\mathcal{F} \in \mathcal{O}_X$-Mod, write $\overline{\mathcal{F}}$ for the corresponding object in $\mathcal{O}_X\text{-Mod}/C_X$. Set $D_X = C_X \cap \mathcal{O}_X$-mod and $D_{\mathcal{R}} = C_{\mathcal{R}} \cap \mathrm{qgr}\text{-}\mathcal{R}$.

As in Proposition 5.3, define a map $\theta' : \mathcal{O}_X\text{-mod} \to \mathrm{qgr}\text{-}\mathcal{R}$ by $\mathcal{F} \mapsto \pi(\mathcal{F}\otimes\mathcal{R})$. By that proposition, $\theta'(k(c_i))$ is a direct sum of copies of $\widetilde{c}_i$, and so θ' induces a functor $\theta : \mathcal{O}_X\text{-mod}/D_X \to \mathrm{qgr}\text{-}\mathcal{R}/D_{\mathcal{R}}$. Conversely, let $\mathcal{M} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in \mathrm{gr}\text{-}\mathcal{R}$. Then Lemma 6.3 implies that, for some ω,

$$\mathcal{F}_n \otimes \mathcal{I}^{\sigma^n} \cong \mathcal{F}_n\mathcal{I}^{\sigma^n} \cong \mathcal{F}_{n+1} \qquad \text{for } n \geq \omega. \tag{6.10}$$

Thus, $\overline{\mathcal{F}}_n \cong \overline{\mathcal{F}}_{n+1}$ for all $n \geq \omega$ and so the rule $\psi' : \mathcal{M} \mapsto \overline{\mathcal{F}}_\omega$ defines a functor from qgr-$\mathcal{R}$ to $\mathcal{O}_X\text{-mod}/D_X$. For all i, the object $\widetilde{c}_i$ maps to 0 and so there is an induced functor $\psi : \mathrm{qgr}\text{-}\mathcal{R}/D_{\mathcal{R}} \to \mathcal{O}_X\text{-mod}/D_X$.

We need to check that these are inverse functors. First, since $\mathcal{I}_n$ and $\mathcal{O}_X$ are isomorphic modulo D_X, it is clear that $\psi'\theta'(\mathcal{F}) = \overline{\mathcal{F}}$ for $\mathcal{F} \in \mathcal{O}_X$-mod. On the other hand, if $\mathcal{M} \in \mathrm{gr}\text{-}\mathcal{R}$ and ω satisfy (6.10), then $\theta\psi'(\mathcal{M})$ is the image of the module $\mathcal{M}' = \mathcal{F}_\omega \otimes \mathcal{R} \in \mathrm{gr}\text{-}\mathcal{R}$. Therefore, $\mathcal{M}'_{\geq\omega} = \mathcal{F}_\omega \otimes \mathcal{I}_\omega \otimes \mathcal{L}_\sigma^{\otimes\omega} \otimes \mathcal{R}$ whereas $\mathcal{M}_{\geq\omega} = \mathcal{F}_\omega \otimes \mathcal{L}_\sigma^{\otimes\omega} \otimes \mathcal{R}$, by the choice of ω. Thus, there is a natural map $\alpha : \mathcal{M}'_{\geq\omega} \to \mathcal{M}_{\geq\omega}$ whose cokernel is a homomorphic image of $\mathcal{F}_\omega \otimes \mathcal{O}_X/\mathcal{I}_\omega \otimes \mathcal{L}_\sigma^{\otimes\omega} \otimes \mathcal{R}$, which is clearly contained in $\theta'(D_X) \subseteq D_{\mathcal{R}}$. Similarly, Lemma 3.3 implies that $\mathrm{Ker}(\alpha) \subseteq D_{\mathcal{R}}$ and so $\mathcal{M} = \mathcal{M}'$ in $\mathrm{gr}\text{-}\mathcal{R}/D_{\mathcal{R}}$.

Therefore, both $\theta\psi$ and $\psi\theta$ are naturally isomorphic to the identity and we have an equivalence $\mathcal{O}_X\text{-mod}/D_X \simeq \mathrm{qgr}\text{-}\mathcal{R}/D_{\mathcal{R}}$. It now follows formally, as in the proof of Theorem 6.7, that this induces the desired equivalence $\mathcal{O}_X\text{-Mod}/C_X \simeq \mathrm{Qgr}\text{-}\mathcal{R}/C_{\mathcal{R}}$. $\square$

7. The chi conditions

The hypotheses from Assumptions 4.9 remain in force throughout the current section. The aim is to prove that $R = R(X, c, \mathcal{L}, \sigma)$ satisfies the χ_1 condition but not the χ_2 condition, as defined in the introduction. In the process we will prove that $\mathrm{H}^1(\pi(R)) = \mathrm{Ext}^1_{\mathrm{Qgr}\text{-}R}(\pi(R), \pi(R))$ is infinite dimensional, thereby proving parts 7 and 8 of Theorem 1.1. An easy consequence (see Corollary 7.11) will be that qgr-R satisfies the ampleness condition of Artin and Zhang [AZ1]. For a detailed discussion of these various conditions the reader is referred to [AZ1, SV].

Theorem 7.1. *Keep the hypotheses from* (4.9). *Then, on both the left and the right:*

1. χ_1 *holds for* R.
2. χ_2 *fails for* R. *Indeed* $\mathrm{Ext}^2_{\text{Mod-}R}(k,R)$ *is infinite dimensional.*
3. $\mathrm{H}^1(\pi(R)) = \mathrm{Ext}^1_{\text{Qgr-}R}(\pi(R),\pi(R))$ *is infinite dimensional.*

Before beginning the proof of the theorem we need two easy technical lemmas. The proofs will use the derived functors $\mathcal{E}xt^i$ of $\mathcal{H}om$, the Sheaf Hom on $\mathcal{O}_X$-modules in the sense of [Ha, Section III.6].

Lemma 7.2. *Let* $\mathcal{F} \in \mathcal{O}_X\text{-mod}$ *and fix* $i \in \mathbb{Z}$. *Then:*

1. *If* $c_i \notin \operatorname{supp}\mathcal{F}$, *then* $\mathrm{Hom}_{\mathcal{O}_X}(k(c_i),\mathcal{F}) = 0 = \mathrm{Ext}^1_{\mathcal{O}_X}(k(c_i),\mathcal{F})$.
2. *Suppose that* $\mathcal{F} \subseteq \mathcal{M}$, *where* $\mathcal{M}$ *is locally free with* $c_i \notin \operatorname{supp}\mathcal{M}/\mathcal{F}$. *Then* $\mathrm{Ext}^1_{\mathcal{O}_X}(k(c_i),\mathcal{F}) = 0$.
3. *If* $\mathcal{F} \in \mathrm{GT}\,\mathcal{O}_X\text{-mod}$ *then* $\mathrm{Ext}^j_{\mathcal{O}_X}(\mathcal{I}_n,\mathcal{F}) = 0$ *for all* $j \geq \dim X$ *and all* $n \geq 0$.

Proof. (1,2) Let $\mathcal{O}(1)$ be a very ample invertible sheaf on X. Then $k(c_i)\otimes\mathcal{O}(n) \cong k(c_i)$ for all $n \in \mathbb{Z}$ and so, by [Ha, Propositions III.6.7 and III.6.9], it suffices to prove that $\mathcal{E}xt^j(k(c_i),\mathcal{F}) = 0$ (where $j = 0, 1$ in part 1 and $j = 1$ in part 2). By [Ha, Proposition III.6.8] the question is now a local one; for any $x \in X$ one has $\mathcal{E}xt^i(k(c_i),\mathcal{F})_x = \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(k(c_i)_x,\,\mathcal{F}_x)$.

Part 1 now follows from the fact that, for any closed point $x \in X$, either $k(c_i)$ or $\mathcal{F}$ is zero at x. Hence so are the $\mathcal{E}xt$ groups.

In order to prove part 2, note that, by part 1, $\mathcal{H}om(k(c_i),\mathcal{M}/\mathcal{F}) = 0$. So by the long exact sequence in $\mathcal{E}xt$ it is enough to show that $\mathcal{E}xt^1(k(c_i),\mathcal{M}) = 0$. But $\dim X \geq 2$ and, by Lemma 3.7, X is smooth and hence Cohen-Macaulay at c_i. Thus $\mathcal{E}xt^1(k(c_i),\mathcal{M})_{c_i} = 0$. At any other closed point $c_i \neq y \in X$, one has $k(c_i)_y = 0$ and hence $\mathcal{E}xt^1(k(c_i),\mathcal{M})_y = 0$.

(3) Pick $j \geq d = \dim X$ and consider the exact sequence

$$\mathrm{Ext}^j(\mathcal{O}_X,\mathcal{F}) \to \mathrm{Ext}^j(\mathcal{I}_n,\mathcal{F}) \to \mathrm{Ext}^{j+1}\Big(\bigoplus_{i=0}^{n-1} k(c_i),\mathcal{F}\Big).$$

Since $\mathcal{F}$ is torsion, $\mathcal{F}$ is supported on a subscheme of dimension $< d$ and so $\mathrm{Ext}^j(\mathcal{O}_X,\mathcal{F}) = 0$, by [Ha, Lemma III.2.10]. It therefore suffices to show that $\mathrm{Ext}^m(k(c_i),\mathcal{F}) = 0$, for all $m > d$. It again suffices to prove this locally, and since $k(c_i)$ is supported at c_i we only need to look locally at c_i. Since X is smooth at c_i, $\mathcal{O}_{X,c_i}$ has global dimension $\leq d$ and hence $\mathcal{E}xt^m(k(c_i),\mathcal{F})_{c_i} = 0$. $\square$

Lemma 7.3. *As an* $\mathcal{O}_X$*-module,* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_n,\mathcal{O}_X/\mathcal{I}_n) \cong \bigoplus_{i=0}^{n-1} k(c_i)^d$, *where* $d = \dim X$. *Thus* $\dim_k \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n,\mathcal{O}_X/\mathcal{I}_n) = nd$.

Proof. Looking locally, one calculates that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_n,\mathcal{O}_X/\mathcal{I}_n) \cong \mathcal{I}_n/\mathcal{I}_n^2$. But $\mathcal{O}_X/\mathcal{I}_n \cong \bigoplus_{i=0}^{n-1} k(c_i)$. Since X is smooth locally at each point c_i, the first statement follows. The second assertion follows because $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n,\mathcal{O}_X/\mathcal{I}_n)$ is isomorphic to the global sections of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_n,\mathcal{O}_X/\mathcal{I}_n)$. $\square$

Proof of Theorem 7.1. We prove this result on the right only; the left-sided results then follow by appealing to (3.2).

(1) We begin by making some reductions to the problem. One characterization of the χ_1 condition for R is that, for all $N \in \text{gr-}R$, the natural map

$$N \to \bigoplus_m \mathrm{Hom}_{\mathrm{Qgr}\text{-}R}(\pi(R), \pi(N)[m]) \tag{7.4}$$

should have right bounded cokernel [AZ1, Proposition 3.14]. This condition clearly holds for N if and only if it holds for a shift $N[r]$. Since N has a filtration by shifts of R and Goldie torsion modules, it suffices to prove the condition in those two cases.

We convert (7.4) into a statement about the $\mathcal{R}$-module $\mathcal{N} = (N \otimes_R \mathcal{R})$. By the equivalences of categories, Theorem 4.1, $\xi^{-1} \circ \pi_R(N) = \pi_{\mathcal{R}}(\mathcal{N})$ and $N = \Gamma(\mathcal{N})$ in high degree. Thus we may rephrase χ_1 as requiring that the natural map

$$\Gamma(\mathcal{N}) \to \bigoplus_m \mathrm{Hom}_{\mathrm{Qgr}\text{-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{N})[m]) \tag{7.5}$$

has a right bounded cokernel. Since $R \otimes_R \mathcal{R} = \mathcal{R}$ and $N \otimes_R \mathcal{R}$ is Goldie torsion for any Goldie torsion module N, it will also suffice to show that (7.5) has right bounded cokernel when $\mathcal{N}$ is either $\mathcal{R}$ or a Goldie torsion module.

Write $\mathcal{N} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ and $\mathcal{R} = \bigoplus \mathcal{I}_n \otimes \mathcal{L}_\sigma^{\otimes n}$. Fix $m \gg 0$ and write $\mathcal{N}[m] = \bigoplus \mathcal{G}_n \otimes \mathcal{L}_\sigma^{\otimes n}$; thus $\mathcal{G}_n \cong (\mathcal{F}_{n+m} \otimes \mathcal{L}_m)^{\sigma^{-m}}$ by Lemma 5.5. By Lemma 6.4(1) we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Qgr}\text{-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{N})[m]) \cong \varinjlim_{n\to\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{G}_n).$$

Write

$$\psi_n : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{G}_n) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{n+1}, \mathcal{G}_{n+1}) \tag{7.6}$$

for the n^{th} map in this direct limit and observe that the zeroth term is nothing more than $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_0, \mathcal{G}_0) = \mathrm{H}^0(X, \mathcal{G}_0) \cong \mathrm{H}^0(X, \mathcal{F}_m \otimes \mathcal{L}_m) = \Gamma(\mathcal{N})_m$. Thus if we can show, for $m \gg 0$, that the map ψ_n is an isomorphism for all $n \geq 0$, then we will have shown that (7.5) has right bounded cokernel and proved the result.

As m is sufficiently large, Lemma 6.3 implies that $\mathcal{G}_{n+1} = \mathcal{G}_n \mathcal{I}^{\sigma^n}$ for all $n \geq 0$. In particular, $\mathcal{I}_{n+1} \subseteq \mathcal{I}_n$, $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$, and so Lemma 6.4(1) implies that ψ_n is just the restriction map for all $n \geq 0$. It is clear then that ψ_n fits into the following commutative diagram, where the rows and columns are parts of the long exact sequences in Hom induced from the aforementioned inclusions:

$$\begin{tikzcd}
\mathrm{Hom}(k(c_n), \mathcal{G}_{n+1}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{I}_n, \mathcal{G}_{n+1}) \arrow[r, "\theta_{n+1}"] \arrow[d] & \mathrm{Hom}(\mathcal{I}_{n+1}, \mathcal{G}_{n+1}) \arrow[r] \arrow[d, "\rho_n"] & \mathrm{Ext}^1(k(c_n), \mathcal{G}_{n+1}) \arrow[d] \\
\mathrm{Hom}(k(c_n), \mathcal{G}_n) \arrow[r] & \mathrm{Hom}(\mathcal{I}_n, \mathcal{G}_n) \arrow[r, "\theta_n"'] \arrow[ur, "\psi_n"'] & \mathrm{Hom}(\mathcal{I}_{n+1}, \mathcal{G}_n) \arrow[r] & \mathrm{Ext}^1(k(c_n), \mathcal{G}_n)
\end{tikzcd}$$

Assume first that $\mathcal{N} = \mathcal{R}$ and fix $n \geq 0$. Then $\mathcal{G}_n = (\mathcal{I}_{n+m} \otimes \mathcal{L}_m)^{\sigma^{-m}}$, and so $\mathrm{supp}(\mathcal{L}_m^{\sigma^{-m}}/\mathcal{G}_n) = \{c_{-m}, \ldots, c_{n-1}\} \not\ni c_n$. By Lemma 7.2(2) this implies that $\mathrm{Ext}^1(k(c_n), \mathcal{G}_n) = 0$. Also $\mathrm{Hom}(k(c_n), \mathcal{G}_n) = 0$ since $\mathcal{G}_n$ is a subsheaf of an invertible sheaf. Thus θ_n is an isomorphism. Since ρ_n is injective, the maps ψ_n and ρ_n are also isomorphisms for all $n \geq 0$. For later reference, note that this step works for any $m \geq 0$.

Now assume that $\mathcal{N}$ is Goldie torsion. For $n \geq n_0 \gg 0$, Lemma 6.1 implies that $\mathcal{F}_n = \mathcal{F}_{n+1} = \mathcal{F}$, say. We may assume that $m \geq n_0$ and hence that $\mathcal{G}_n = \mathcal{G}_{n+1} =$

$\mathcal{G} = (\mathcal{F} \otimes \mathcal{L}_m)^{\sigma^{-m}}$ for all $n \geq 0$. Since $\mathcal{F}$ is a Goldie torsion sheaf, its support is a proper closed subset of X and so $\operatorname{supp} \mathcal{F} \cap \{c_i\}_{i \in \mathbb{Z}}$ is finite, say contained in $\{c_j : |j| \leq r\}$. Thus $\operatorname{supp} \mathcal{G} \cap \{c_i\}_{i \in \mathbb{Z}} \subseteq \{c_j : j \leq r - m\}$. As m is sufficiently large we may also assume that $m > r$ and hence that $\operatorname{supp} \mathcal{G} \cap \{c_i : i \geq 0\} = \emptyset$. Lemma 7.2(1) therefore implies that $\operatorname{Hom}(k(c_n), \mathcal{G}) = 0 = \operatorname{Ext}^1(k(c_n), \mathcal{G})$ for all $n \geq 0$. The commutative diagram is now just the statement that $\psi_n = \theta_n = \theta_{n+1}$ is an isomorphism for all $n \geq 0$.

Thus, in either case ψ_n is an isomorphism for $n \geq 0$ and so (7.5) does have the required bounded cokernel.

(2) In order to prove that χ_2 fails on the right for the R-module R, it suffices to show that $\dim_k \operatorname{Ext}^1_{\text{Qgr-}R}(\pi(R), \pi(R)) = \infty$ [AZ1, Theorem 7.4]. Thus (2) follows from (3).

(3) By Theorem 4.1, again, this is equivalent to showing that

$$\dim_k \operatorname{Ext}^1_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{R})) = \infty.$$

Write $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{L}_\sigma^{\otimes n}$, which we think of as a right $\mathcal{R}$-module. The long exact sequence in Hom induced from the inclusion $\pi(\mathcal{R}) \subset \pi(\mathcal{B})$ provides the following exact sequence:

$$\begin{aligned} \text{(7.7)} \qquad \operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{B})) &\xrightarrow{\phi_1} \operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{B}/\mathcal{R})) \\ &\xrightarrow{\phi_2} \operatorname{Ext}^1_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{R})). \end{aligned}$$

We need to understand the first two terms in the sequence.

Lemma 6.4(1) implies that $\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{B})) \cong \lim_{n\to\infty} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{O}_X)$. By Lemma 7.2(2), $\operatorname{Ext}^1(k(c_i), \mathcal{O}_X) = 0$ and so the natural map $\operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X) \to \operatorname{Hom}(\mathcal{I}_n, \mathcal{O}_X)$ is an isomorphism for all $n \geq 0$. Thus $\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{B})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = k$.

On the other hand, $\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{B}/\mathcal{R})) \cong \lim_{n\to\infty} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{O}_X/\mathcal{I}_n)$. As $\operatorname{Hom}_{\mathcal{O}_X}(k(c_n), \mathcal{O}_X/\mathcal{I}_n) = 0$, it follows from the same commutative diagram as in part 1 that all of the maps ψ_n in this direct limit are injective. However, by Lemma 7.3, $\lim_{n\to\infty} \dim_k \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{O}_X/\mathcal{I}_n) = \infty$ and so

$$\dim_k \lim_{n\to\infty} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{O}_X/\mathcal{I}_n) = \infty.$$

Thus the map ϕ_2 in (7.7) has an infinite dimensional cokernel, and we are done. $\square$

In the proof of Theorem 7.1 we were careful to show that, for $\mathcal{N} = \mathcal{R}$, (7.6) is an isomorphism for all $m \geq 0$ and hence that (7.5) is an isomorphism. This gives the following corollary.

Corollary 7.8. $R \cong \bigoplus_{m \geq 0} \operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{R}), \pi(\mathcal{R})[m])$. $\square$

One of the main aims of [AZ1] was to show that, up to a finite dimensional vector space, one can recover a connected graded noetherian ring S from qgr-S whenever S satisfies χ_1. Conversely, given an appropriate category $\mathcal{C}$ with an ample shift functor then one can construct a "ring of sections" S with qgr-$S \simeq \mathcal{C}$. It is almost immediate from Theorem 7.1 that these results do apply to R. The reader should note, however, that the definitions of ampleness and rings of sections from [AZ1] are not the same as the ones used in this paper. Instead, they are defined as follows:

Definition 7.9. Let S be a noetherian connected graded k-algebra. Then the shift functor $s : \mathcal{P} \to \mathcal{P}[1]$ on qgr-S is called *Artin-Zhang ample* provided:

(a) For all $\mathcal{M} \in$ qgr-S there is an epimorphism $\bigoplus_{i=0}^{m} \pi(S)[-\ell_i] \twoheadrightarrow \mathcal{M}$ for some $\ell_i \geq 0$.
(b) For every epimorphism $f : \mathcal{M} \twoheadrightarrow \mathcal{N}$ in qgr-S, and for all $n \gg 0$, the induced map $\mathrm{Hom}_{\text{qgr-}S}(\pi(S), \mathcal{M}[n]) \to \mathrm{Hom}_{\text{qgr-}S}(\pi(S), \mathcal{N}[n])$ is surjective.

One also has a natural functor, which we write as Γ_{AZ} to distinguish it from the global sections functor Γ, from qgr-$S \to$ Gr-S defined by

$$\Gamma_{AZ}(\mathcal{M}) = \bigoplus_{n \geq 0} \mathrm{Hom}_{\text{qgr-}S}(\pi(S), \mathcal{M}[n]). \tag{7.10}$$

Combining the results of this section with those from [AZ1] we obtain:

Corollary 7.11. *Keep the hypotheses from* (4.9)*. Then:*

(1) $\Gamma_{AZ}(\pi_R(R)) = R$.
(2) *The shift functor s is Artin-Zhang ample in* qgr-R.

Proof. (1) Using Theorem 4.1, this is just a restatement of Corollary 7.8.
(2) By [AZ1, Theorem 4.5] and part 1, this is equivalent to R satisfying χ_1. $\square$

As a final application of Theorem 7.1 we note that, by [YZ, Theorem 4.2], Theorem 7.1 implies that R does not have a balanced dualizing complex, in the sense of Yekutieli [Ye]. By [Jg] and Theorem 8.2, below, it does however have a dualizing complex in the weaker sense of [Jg].

8. Homological and cohomological dimensions

The hypotheses of Assumptions 4.9 remain in force in this section. We will continue to study the homological properties of Qgr-$R \simeq$ Qgr-$\mathcal{R}$ by showing that this category always has finite cohomological dimension and even has finite homological dimension when X is smooth. This proves part 6 of Theorem 1.1 from the introduction. In some sense this result is not surprising: it is not difficult to reduce both results to a consideration of Goldie torsion modules, in which case Theorem 6.7 can be applied.

We recall the definitions of these concepts:

Definition 8.1. The *global dimension* of Qgr-$\mathcal{R}$ (or Qgr-R) is defined to be

$$\mathrm{gld}(\mathcal{R}) = \sup\{i \mid \mathrm{Ext}^i_{\text{Qgr-}\mathcal{R}}(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{M}, \mathcal{N} \in \text{Qgr-}\mathcal{R}\}.$$

The *cohomological dimension* of Qgr-$\mathcal{R}$ (or Qgr-R) is defined to be

$$\mathrm{cd}(\text{Qgr-}\mathcal{R}) = \sup\{\mathrm{cd}(\mathcal{N}) \ \mid \ \mathcal{N} \in \text{Qgr-}\mathcal{R}\},$$

where $\mathrm{cd}(\mathcal{N}) = \sup\{i \mid \mathrm{H}^i(\mathcal{N}) \neq 0.\}$

Cohomological dimension is just defined for qgr-R in [AZ1] but, by [AZ1, Proposition 7.2(4)], this is equivalent to our definition.

Theorem 8.2. *Keep the hypotheses from* (4.9)*. Then* Qgr-$\mathcal{R}$ *has finite cohomological dimension. Indeed,* $\dim X - 1 \leq \mathrm{cd}(\text{Qgr-}\mathcal{R}) \leq \dim X$.

Proof. In order to give the upper bound, what we need to prove is that $\mathrm{H}^i(\mathcal{M}) = \mathrm{Ext}^i_{\mathrm{Qgr}\text{-}\mathcal{R}}(\pi(\mathcal{R}), \mathcal{M}) = 0$ for all $i > d = \dim X \geq 2$ and all $\mathcal{M} \in \mathrm{Qgr}\text{-}\mathcal{R}$. We will drop π in the proof. Let $\mathcal{E}$ denote the injective hull of $\mathcal{M}$ in Qgr-$\mathcal{R}$. By Theorem 4.1 and [AZ1, (7.1.4)], injective hulls in $\mathrm{Qgr}\text{-}\mathcal{R} \simeq \mathrm{Qgr}\text{-}R$ are induced from those in Gr-R. This implies that $\mathcal{E}/\mathcal{M} \in \mathrm{GT}\,\mathrm{Qgr}\text{-}\mathcal{R}$, since the analogous statement holds in Gr-R. Since $\mathrm{Ext}^i_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{R}, \mathcal{M}) = \mathrm{Ext}^{i-1}_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{R}, \mathcal{E}/\mathcal{M})$, it suffices to prove that $\mathrm{Ext}^j_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{R}, \mathcal{N}) = 0$ for $j \geq \dim X$ and $\mathcal{N} \in \mathrm{GT}\,\mathrm{Qgr}\text{-}\mathcal{R}$. By [AZ1, Proposition 7.2(4)], we may also assume that $\mathcal{N} \in \mathrm{GT}\,\mathrm{qgr}\text{-}\mathcal{R}$.

By Lemma 6.1, we may write $\mathcal{N} = \bigoplus_{n\geq 0} \mathcal{F} \otimes \mathcal{L}_\sigma^{\otimes n}$, for some $\mathcal{F} \in \mathrm{GT}\,\mathcal{O}_X\text{-mod}$. Now take an injective resolution $\mathcal{F} \to \mathcal{E}^\bullet$ of $\mathcal{F}$ in $\mathcal{O}_X$-mod and observe that, since $\mathcal{F} \in \mathrm{GT}\,\mathcal{O}_X\text{-mod}$, so is each $\mathcal{E}^\ell$. Thus, by Theorem 6.7 this induces an injective resolution $\mathcal{N} \to \bigoplus_{n\geq 0} \mathcal{E}^\bullet \otimes \mathcal{L}_\sigma^{\otimes n}$ of $\mathcal{N}$ in Qgr-$\mathcal{R}$. Thus, for all $j \geq \dim X$ we have

$$\begin{aligned}
\mathrm{Ext}^j_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{R}, \mathcal{N}) &= h^j \,\mathrm{Hom}_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{R}, \mathcal{E}^\bullet \otimes \mathcal{L}_\sigma^{\otimes n}) \\
&= h^j \lim_{n\to\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{E}^\bullet) && \text{by Lemma 6.4(1)} \\
&= \lim_{n\to\infty} h^j \,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{E}^\bullet) \\
&= \lim_{n\to\infty} \mathrm{Ext}^j_{\mathcal{O}_X}(\mathcal{I}_n, \mathcal{F}) = 0 && \text{by Lemma 7.2(3).}
\end{aligned}$$

Thus, $\mathrm{cd}(\mathrm{Qgr}\text{-}\mathcal{R}) \leq \dim X$.

For the other direction, by Proposition 3.5 it suffices to prove the result for $\mathcal{R} = \mathcal{R}(X, c, \mathcal{O}_X, \sigma)$. By Lemma 3.7, X is smooth at the point $x = c_{-1}$ and so $\mathcal{O}_{X,x}$ has global dimension equal to $d = \dim X$. Thus $\mathrm{Ext}^d_{\mathcal{O}_{X,x}}(k(x), k(x)) \neq 0$ (use, for example, [Rt, Exercise 9.25 and Corollary 9.55]). Using the arguments at the beginning of the proof of Lemma 7.2 this implies that $\mathrm{Ext}^d_{\mathcal{O}_X}(k(x), k(x)) \neq 0$.

Now notice that $\mathrm{GT}\,\mathcal{O}_X\text{-Mod}$ is a localizing subcategory of $\mathcal{O}_X$-Mod which is closed under essential extensions and hence injective hulls. The same is true of the subcategory $\mathrm{GT}\,\mathrm{Qgr}\text{-}\mathcal{R}$ of Qgr-$\mathcal{R}$. It follows that Ext groups involving only Goldie torsion objects may be calculated entirely within the Goldie torsion subcategories. Then by Theorem 6.7 and Remark 6.8 we have $\mathrm{Ext}^d_{\mathrm{Qgr}\text{-}\mathcal{R}}(\widetilde{x}, \widetilde{x}) \cong \mathrm{Ext}^d_{\mathcal{O}_X}(k(x), k(x)) \neq 0$. Since we have taken $\mathcal{L} = \mathcal{O}_X$, Definition 5.6 and Proposition 5.8 provides a short exact sequence $0 \to \mathcal{R}[+1] \to \mathcal{R} \to \widetilde{x} \to 0$. This induces the exact sequence

$$\mathrm{Ext}^{d-1}_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{R}[+1], \widetilde{x}) \to \mathrm{Ext}^d_{\mathrm{Qgr}\text{-}\mathcal{R}}(\widetilde{x}, \widetilde{x}) \to \mathrm{Ext}^d_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{R}, \widetilde{x}).$$

Since $\widetilde{x}$ is Goldie torsion, $\mathrm{Ext}^d_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{R}, \widetilde{x}) = 0$ by the first part of the proof. Therefore, $\mathrm{H}^{d-1}(\widetilde{x}[-1]) \cong \mathrm{Ext}^{d-1}_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{R}[+1], \widetilde{x}) \neq 0$. □

Corollary 8.3. *If X is smooth, then* Qgr-$\mathcal{R}$ *has finite global dimension. Indeed,* $\dim X \leq \mathrm{gld}(\mathcal{R}) \leq 1 + \dim X$.

Proof. In the proof of Theorem 8.2 we show that $\mathrm{Ext}^d_{\mathrm{Qgr}\text{-}\mathcal{R}}(\widetilde{c}_{-1}, \widetilde{c}_{-1}) \neq 0$, which gives the required lower bound.

In order to prove the upper bound, we need to prove that $\mathrm{Ext}^i_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{M}, \mathcal{N}) = 0$ for all $\mathcal{M}, \mathcal{N} \in \mathrm{Qgr}\text{-}\mathcal{R}$ and all $i \geq \dim X + 1$. The first step is to reduce to the case of noetherian objects. One can assume that $\mathcal{M}$ is noetherian by the usual module-theoretic argument [Rt, Theorem 9.12] provided one replaces Baer's criterion by

[Gr, Lemma 1, p.136]. Now that $\mathcal{M}$ is noetherian, the functor $\mathrm{Ext}^i_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{M},-)$ commutes with direct limits [AZ1, Proposition 7.2(4)] and so we may assume that $\mathcal{N}$ is also noetherian.

So, assume that $\mathcal{M}$ and $\mathcal{N}$ are noetherian. By Lemma 6.2, we may assume that $\mathcal{M}$ is either a shift of $\mathcal{R}$ or a Goldie torsion module. If $\mathcal{M} = \mathcal{R}[r]$, then $\mathrm{Ext}^i_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{M},\mathcal{N}) = \mathrm{Ext}^i_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{R},\mathcal{N}[-r]) = 0$, by Theorem 8.2. Thus we may assume that $\mathcal{M} \in \mathrm{GT}\,\mathrm{qgr}\text{-}\mathcal{R}$.

As in the proof of Theorem 8.2, it suffices to prove that $\mathrm{Ext}^j_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{M},\mathcal{N}) = 0$ for all $\mathcal{N} \in \mathrm{GT}\,\mathrm{qgr}\text{-}\mathcal{R}$ and all $j > \dim X$. By Theorem 6.7, we may write $\mathcal{M} = \bigoplus_{n\geq 0} \mathcal{F}\otimes\mathcal{L}_\sigma^{\otimes n}$ and $\mathcal{N} = \bigoplus_{n\geq 0} \mathcal{G}\otimes\mathcal{L}_\sigma^{\otimes n}$ for some $\mathcal{F},\mathcal{G} \in \mathrm{GT}\,\mathcal{O}_X$-mod. Moreover, the argument given in the proof of Theorem 8.2 shows that we may calculate Ext inside the Goldie torsion subcategories and so $\mathrm{Ext}^j_{\mathrm{Qgr}\text{-}\mathcal{R}}(\mathcal{M},\mathcal{N}) = \mathrm{Ext}^j_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$. As X is smooth, $\mathcal{O}_X$-mod has homological dimension $\dim X$ [Ha, Exercise III.6.5 and Proposition III.6.11A]. Thus $\mathrm{Ext}^j_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) = 0$ for all $j > \dim X$ and $\mathrm{gld}(\mathcal{R}) \leq 1 + \dim X$. □

In both Theorem 8.2 and Corollary 8.3 we conjecture that the correct dimension is $\dim X$.

Curiously, there seems to be no known example of a noetherian connected graded algebra that does *not* have finite cohomological dimension. We presume that such examples do exist, if only because the standard proofs that commutative varieties have finite cohomological dimension clearly do not work in a noncommutative setting.

9. Generic flatness

The hypotheses from Assumptions 4.9 will be assumed throughout this section. Recall that a k-algebra A is *strongly noetherian* if $A\otimes_k C$ is noetherian for all commutative noetherian k-algebras C. The strong noetherian condition was introduced in [ASZ, AZ2] where it is shown that strongly noetherian graded algebras have a remarkable number of nice properties. Notably, [ASZ, Theorem 0.1] shows that they satisfy generic flatness, as defined in the introduction. In this section we show that $R = R(X,c,\mathcal{L},\sigma)$ fails generic flatness in a quite dramatic way: for *any* open affine subset $V \subseteq X$ it fails for the algebras $C = \mathcal{O}_X(V) \subset A = R\otimes_k C$ and the module $M = \mathcal{R}(V)$. We will also use this to construct an explicit noetherian ring C such that $R\otimes_k C$ is not noetherian.

Lemma 9.1. *Let V be an open affine subset of X and set $C = \mathcal{O}_X(V)$. Then $M = \mathcal{R}(V)$ is a finitely generated (C,R)-bimodule. Equivalently, (after identifying C^{op} with C) M is a finitely generated right $R\otimes_k C$-module.*

Proof. Trivially M is a graded left C-module. Write $\mathcal{J}_n = \mathcal{I}_n\otimes\mathcal{L}_n$ for $n\geq 0$; thus $\mathcal{R}_n = {}_1(\mathcal{J}_n)_{\sigma^n}$. Then the right R-module structure on M is the natural one induced by the maps

$$\mathcal{J}_n(V)\otimes\mathcal{J}_m(X) \xrightarrow{1\otimes(\sigma^n)^*} \mathcal{J}_n(V)\otimes\mathcal{J}_m^{\sigma^n}(X) \longrightarrow \mathcal{J}_n(V)\otimes\mathcal{J}_m^{\sigma^n}(V) \longrightarrow \mathcal{J}_{n+m}(V)$$

for $m,n\geq 0$. The commutativity of the two actions is clear.

By Theorem 4.1, there exists n_0 such that $\mathcal{R}_n$ is generated by its sections for all $n\geq n_0$. Thus the natural map $R_{\geq n_0}\otimes C \to M_{\geq n_0}$ is a surjective $R\otimes C$-module homomorphism. Since R is noetherian, $R_{\geq n_0}$ is a finitely generated right R-module

and hence $M_{\geq n_0}$ is a finitely generated $R \otimes C$-module. Finally, as $\bigoplus_{i=0}^{n_0-1} M_i$ is a finitely generated left C-module, M is indeed a finitely generated $R \otimes C$-module. $\square$

Theorem 9.2. *Keep the hypotheses of* (4.9). *Let V be any open affine subset of X and write $C = \mathcal{O}_X(V)$ and $M = \mathcal{R}(V)$. Then M is a finitely generated right $R \otimes_k C$-module which is not generically flat over C.*

Thus, R is neither strongly right noetherian nor strongly left noetherian.

Remarks 9.3. (1) One can make the theorem more precise by exactly determining the maximal ideals $\mathfrak{p}$ of C at which M is not flat. Indeed, let $\mathfrak{m}_x$ denote the maximal ideal of C corresponding to the closed point $x \in X$. Then M fails to be flat at precisely the maximal ideals $\mathfrak{m}_{c_i}$ for $i \in P = \{n \geq 0 : c_n \in V\}$. Moreover, M_t will not be flat at such an $\mathfrak{m}_{c_i}$ whenever $t > i$. Note that, as $\{c_i\}_{i \geq 0}$ is critically dense, $\mathbb{N} \smallsetminus P$ is a finite set.

(2) If one wishes to work more scheme-theoretically, then Theorem 9.2 generalizes naturally to one describing $\mathcal{R}$ as a sheaf of right modules over $R \otimes_k \mathcal{O}_X$.

(3) The theorem proves Theorem 1.1(3) and Proposition 1.2 from the introduction.

Proof. If R is strongly right noetherian, then so is $R \otimes_k C$, since C is a finitely generated k-algebra. Thus once we prove that M is not generically flat over C, it follows from [ASZ, Theorem 0.1] that R cannot be strongly right noetherian. It then follows from (3.2) that R is not strongly left noetherian.

By Lemma 9.1 M is a finitely generated $R \otimes C$-module. Consider the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_n \otimes \mathcal{L}_n \longrightarrow \mathcal{L}_n \longrightarrow \bigoplus_{i=0}^{n-1} k(c_i) \longrightarrow 0.$$

Localizing this sequence at the open set V gives

$$0 \longrightarrow M_n \longrightarrow \mathcal{L}_n(V) \longrightarrow \bigoplus_{i=0}^{n-1} k(c_i)(V) \longrightarrow 0. \tag{9.4}$$

Since $\mathcal{L}_n$ is locally free of rank 1, for any $\mathfrak{p} \in \operatorname{Spec} C$ the module $(\mathcal{L}_n(V))_{\mathfrak{p}}$ will be isomorphic to $C_{\mathfrak{p}}$.

Let P be the set defined by Remark 9.3 and let $\mathfrak{p}$ be any prime ideal in $\operatorname{Spec} C$ that is not equal to $\mathfrak{m}_i = \mathfrak{m}_{c_i}$ for $i \in P$. In particular, $\mathfrak{p} \neq \mathfrak{m}_j$, for $j \geq 0$. Thus, for any n, localizing (9.4) at $\mathfrak{p}$ shows that $(M_n)_{\mathfrak{p}} \cong C_{\mathfrak{p}}$ and hence that $M_{\mathfrak{p}}$ is a flat $C_{\mathfrak{p}}$-module.

On the other hand, if $\mathfrak{p} = \mathfrak{m}_i$ for $i \in P$ and $n - 1 \geq i$, then localizing (9.4) at $\mathfrak{p}$ gives the exact sequence

$$0 \longrightarrow (M_n)_{\mathfrak{m}_i} \longrightarrow C_{\mathfrak{m}_i} \longrightarrow C_{\mathfrak{m}_i}/\mathfrak{m}_i C_{\mathfrak{m}_i} \longrightarrow 0.$$

Thus $(M_n)_{\mathfrak{m}_i} \cong \mathfrak{m}_i C_{\mathfrak{m}_i}$. Suppose that $\mathfrak{m}_i C_{\mathfrak{m}_i}$ is a flat $C_{\mathfrak{m}_i}$-module. Then it is free, and hence principal. By the Krull principal ideal theorem, $\mathfrak{m}_i C_{\mathfrak{m}_i}$ must have height one, contradicting the fact that $\dim C_{\mathfrak{m}_i} \geq 2$. Thus $(M_n)_{\mathfrak{m}_i}$ is never flat for $n \gg 0$ and so $M_{\mathfrak{m}_i}$ is not a flat $C_{\mathfrak{m}_i}$-module when $i \in P$.

Now let $f \in C$ be non-zero. Then $C_f = \mathcal{O}_X(W)$ for some open affine set $W \subseteq V$. Since the conclusion of the last paragraph is independent of the choice of V, this implies that $M_f = \mathcal{R}(W)$ cannot be a flat C_f-module.

Finally, we have shown that $M_{\mathfrak{p}}$ is a flat $C_{\mathfrak{p}}$-module if and only if $\mathfrak{p} \neq \mathfrak{m}_i$ for $i \in P$. This justifies the assertions in Remark 9.3(1). $\square$

A natural question raised by Theorem 9.2 is to identify a commutative noetherian ring C for which $R \otimes C$ is not noetherian. This was achieved in [Ro] for the special case $X = \mathbb{P}^t$, but without an explanation for why the particular ring C given there, an infinite affine blowup of affine space, was a natural choice. We show next how the results of [ASZ] lead inevitably to similar choices of C in general. To set this up, we need to discuss some commutative constructions from [ASZ].

Let C be a finitely generated commutative k-algebra that is a domain with fraction field F, and let c be a closed smooth point of $\operatorname{Spec} C$ with associated maximal ideal $\mathfrak{p}$. The *affine blowup* of $\operatorname{Spec} C$ at c is $\operatorname{Spec} C'$, where the ring C' is formed as follows: write $\mathfrak{p} = \sum_{i=0}^{r} x_i C$ with $x_0 \notin \mathfrak{p}^2$ and define $C' = C[x_1 x_0^{-1}, x_2 x_0^{-1}, \dots, x_r x_0^{-1}] \subset F$. Here x_0 will be called the denominator for the blowup. Now suppose we are given an infinite sequence of distinct smooth closed points $c_1, c_2, \dots$ of $\operatorname{Spec} C$ corresponding to maximal ideals $\mathfrak{p}_1, \mathfrak{p}_2, \dots$, and for each $i \geq 1$ suppose that we can find a denominator $x^{(i)} = x_0^{(i)}$ such that

$$x^{(i)} \in \mathfrak{p}_i \setminus {\mathfrak{p}_i}^2 \text{ but } x^{(i)} \notin \mathfrak{p}_k \text{ for } k \neq i. \tag{9.5}$$

Then we can successively blow up each of the points c_i, giving a sequence of rings $C \subseteq C_1 \subseteq C_2 \subseteq \dots$ The ring $\widetilde{C} = \bigcup C_i$ is called the *infinite affine blowup* of C at the points c_i (with respect to the particular choice of denominators $x^{(i)}$). Let $\rho : \operatorname{Spec} \widetilde{C} \to \operatorname{Spec} C$ be the blowup map, induced by the inclusion $C \to \widetilde{C}$. For each $i \geq 1$ the ideal $\eta_i = \mathfrak{p}_i \widetilde{C}$ is a prime ideal of $\widetilde{C}$ that corresponds to the exceptional divisor $\rho^{-1}(c_i)$ of $\operatorname{Spec} \widetilde{C}$ [ASZ, Lemma 1.3].

Returning to our specific scheme X, let V be an open affine subset of X and set $C = \mathcal{O}_X(V)$. Define P by Remark 9.3 and write $\mathfrak{p}_i = \mathfrak{m}_i$ for the maximal ideal of C corresponding to the point c_i for $i \in P$. Recall that the c_i are smooth by Lemma 3.7. Because the sequence $\{c_i : i \in P\}$ is critically dense, it is possible to choose an infinite subsequence $\mathcal{C}' = \{c_i : i \in I \subseteq P\}$ which have denominators satisfying (9.5) [ASZ, Proposition 1.6], and so the infinite blowup $\widetilde{C}$ of C at that subsequence $\mathcal{C}'$ is well defined. By [ASZ, Theorem 1.5] $\widetilde{C}$ is noetherian. If we invert $\{f \in \widetilde{C} \smallsetminus \bigcup_{i \in I} \mathfrak{p}_i \widetilde{C}\}$, we obtain a further localization D of $\widetilde{C}$ that is a Dedekind domain [ASZ, Proposition 2.8].

Theorem 9.6. *Assume that* (4.9) *holds and set $R = R(X, c, \mathcal{L}, \sigma)$ and let $\widetilde{C}$ and D be defined as above. Then $R \otimes_k \widetilde{C}$ and $R \otimes_k D$ are not noetherian rings.*

Proof. Set $M = \mathcal{R}(V)$. By Remark 9.3(1) $M_{\mathfrak{p}}$ is not flat for exactly the maximal ideals $\mathfrak{p}_i \in C$ which correspond to the points in $\{c_i : i \in P\}$. Following the proof of [ASZ, Theorem 2.3], we see that D has maximal ideals $\{\mu_j = \mathfrak{p}_j D : j \in I\}$ and that $M \otimes_C D_{\mu_j}$ is never a flat D_{μ_j}-module. Since each D_{μ_j} is a DVR, this means that $M \otimes_C D_{\mu_j}$ is not even a torsion-free D_{μ_j}-module [ASZ, Lemma 3.3(4)]. Now the points μ_j are critically dense in $\operatorname{Spec} D$. So if $M \otimes_C D$ were a noetherian $R \otimes_k D$-module, then [ASZ, Lemma 3.3(2)] would imply that $M \otimes_C D$ would be torsion-free at all but finitely many closed points of $\operatorname{Spec} D$. This contradicts our construction and proves that $M \otimes_C D$ is not noetherian. However it is a finitely generated module over $R \otimes_k D \cong R \otimes_k C \otimes_C D$ simply because M is a finitely

generated $R \otimes_k C$-module. Thus $R \otimes_k D$ is not noetherian which, since D is a localization of $\widetilde{C}$, implies that $R \otimes_k \widetilde{C}$ is also not noetherian. $\square$

10. Point modules

The hypotheses from Assumptions 4.9 will remain in force throughout the section. Here we discuss the point modules for $R = R(X, c, \mathcal{L}, \sigma)$ and show in particular that they are not parametrized by a projective scheme and that the shift functor does not induce an automorphism on the set of point modules. This is in marked contrast to the behaviour of strongly noetherian rings and so provides further proofs of the fact that R is not strongly noetherian.

To set this in context, we begin by reviewing the work of [AZ2] and answering a special case of one their conjectures. Let $S = \bigoplus_{n\geq 0} S_n$ be a connected graded k-algebra. Given a commutative k-algebra C, write $S_C = S \otimes_k C = \bigoplus(S_n \otimes_k C)$, regarded as a graded C-algebra. Fix a finitely generated graded S-module P and a Hilbert function $h : \mathbb{N} \to \mathbb{N}$. For a finitely generated commutative k-algebra C, write $\mathcal{P}_h(C)$ for the set of isomorphism classes of graded factors V of $P_C = P \otimes_S S_C$ with the property that each V_n is a flat C-module of constant rank $h(n)$. Given a map of finitely generated algebras $C \to D$, one gets a map from $\mathcal{P}_h(C)$ to $\mathcal{P}_h(D)$ via $M \mapsto M \otimes_C D$. Following [AZ2, Section E4], for an arbitrary commutative k-algebra C we let $\mathcal{P}_h(C)$ denote the direct limit of the sets $\mathcal{P}_h(C')$ as C' varies over the finitely generated subalgebras of C. In this way $\mathcal{P}_h$ becomes a functor from (Rings) to (Sets), where (Rings) denotes the category of commutative k-algebras.

One of the main aims of [AZ2] was to show that for any h the functor $\mathcal{P}_h$ can be represented by a projective scheme. The relevant definitions are just like the commutative case: given a scheme Y over k, its *functor of points* h_Y : (Rings) $\to$ (Sets) is defined by $C \mapsto \mathrm{Morph}(\operatorname{Spec} C, Y)$, where Morph denotes morphisms in the category of k-schemes. By Yoneda's lemma this induces an embedding of the category (Schemes) as a full subcategory of the category (Fun) of covariant functors from (Rings) to (Sets) (see [EH, Section VI] for more details.) A functor $F \in$ (Fun) that is equal to h_Y for some scheme Y is said to be *represented* by Y.

For strongly noetherian algebras one has the following result:

Theorem 10.1. ([AZ2, Theorem E4.3]) *Let S be a strongly noetherian connected graded k-algebra and fix a finitely generated graded S-module P and a Hilbert function h. Then the functor $\mathcal{P}_h$ is represented by a projective scheme Y.*

The paper [AZ2] also proves an analogous but weaker result for parametrizing objects in qgr-S with a given Hilbert series. The definitions are as follows: Given P and h as above, for any finitely generated commutative k-algebra C we let $\mathcal{P}_h^{\mathrm{qgr}}(C)$ denote the set of isomorphism classes of graded factors V of P_C with the property that V_n is a flat C-module of constant rank $h(n)$ for all $n \gg 0$. As before, the definition is extended to all $C \in$ (Rings) by taking limits, and thus $\mathcal{P}_h^{\mathrm{qgr}}$ becomes a functor from (Rings) $\to$ (Sets).[1]

[1]It is not clear what is the best version of flatness to use in this definition. Specifically, [AZ2] uses the formally weaker notion that V be flat in Qgr-S_C rather than requiring that the V_n be flat. The two notions coincide for strongly noetherian algebras by [AZ2, Lemma E5.3]. Fortunately, the distinction is not significant since we will only need to make computations when C is a noetherian domain. In this case both notions of flatness follow automatically from the fact that the V_n have constant rank.

By [AZ2, Theorem E5.1], if S satisfies the *strong χ condition* [AZ2, p.346], then the functor $\mathcal{P}_h^{\mathrm{qgr}}$ is represented by a scheme Y that is a countable union of projective closed subschemes. Artin and Zhang conjecture that Y is actually a projective scheme and the first result of the section shows that this conjecture is true for point modules. These are defined as follows. Suppose that S is a connected graded ring, generated in degree one, $P = S$ and h is the function 1. Then we will write $\mathcal{S}_{\mathrm{pt}}$ for $\mathcal{P}_h$ and $\mathcal{S}_{\text{q-pt}}$ for $\mathcal{P}_h^{\mathrm{qgr}}$. Elements of the set $\mathcal{S}_{\mathrm{pt}}(C)$ will be called *point modules over S_C* while elements of $\mathcal{S}_{\text{q-pt}}(C)$ will be called *point modules in* qgr-S_C. If h is the function $h(t) = 1$ for $0 \le t \le d-1$ but $h(t) = 0$ for $t \ge d$, then the modules in $\mathcal{P}_h(C)$ are defined to be the *truncated point modules of length* d over C.

By [AZ2, Corollary E4.4], if S is strongly noetherian then there exists an integer t (independent of the commutative k-algebra C) such that:

(i) *For all C and $V \in \mathcal{S}_{\mathrm{pt}}(C)$, $\mathrm{Ker}(S \to V)$ is generated in degrees $\le t$.*
(ii) *Conversely, given a truncated point module V' of length $d \ge t$, then V' is the homomorphic image of a unique point module V.*

The significance of these results is that it is easy to show that the truncated point modules of length d are parametrized by a projective scheme Y_d (see [AZ2, Lemma E4.6]). It then follows from (i) and (ii) that, for $d \ge t$, the scheme $Y_d \cong Y_{d+1}$ parametrizes the point modules. As we next show, this scheme also represents $\mathcal{S}_{\text{q-pt}}$.

This result was proved jointly by Michael Artin and the third-named author and we are grateful to Artin for letting us include it here.

Proposition 10.2. *Let S be a noetherian, connected graded k-algebra that is generated in degree one. Assume that S is strongly noetherian or, more generally, that (i) and (ii) hold on both the left and right so that the left, respectively right, point modules are parametrized by a projective scheme Y^ℓ, respectively Y^r. Then:*

(1) *For any commutative noetherian k-algebra C the shift functor $s : M \mapsto M[1]_{\ge 0}$ is an automorphism of the set of right S_C-point modules.*
(2) *The shift functor s induces an automorphism of both Y^r and Y^ℓ.*
(3) *The scheme Y^r also represents the functor $\mathcal{S}_{\text{q-pt}}$.*

Proof. (1) As the base ring C is fixed, we may write S for S_C without confusion. By (i) and (ii), every truncated right S-point module $M = \bigoplus_{i=0}^{r-1} M_i$ of length $r \ge t$ is the factor $N/N_{\ge r}$ for a unique point module N. Thus, to prove the result it suffices to show that there exists a unique shifted truncated S-point module $M' = \bigoplus_{i=-1}^{r-1} M'_i$ such that $M = M'_{\ge 0}$.

As C is noetherian, each M_i is a projective C-module of constant rank one. Consider the Matlis dual $M^\vee = \bigoplus_{i=1-r}^{0} M_i^\vee$ of M; thus $M_i^\vee = \mathrm{Hom}_C(M_{-i}, C)$ for each i. Clearly $M^\vee$ is a left S-module for which each $M_i^\vee$ is a projective C-module of constant rank one. We claim that $M_{1-i}^\vee = S_1 M_{-i}^\vee$ for $1 \le i \le r-1$. It suffices to prove this locally, so assume that C is local. Then each M_i is free, say $M_i = m_i C \cong C$. As S is generated in degree one, $M_i = M_{i-1}S_1 = m_{i-1}S_1$ and $m_i = m_{i-1}s$ for some $s \in S_1$. Let θ be the generator of $M_{-i}^\vee$; thus $m_i^\theta = 1$. Then $\phi = s\theta \in M_{1-i}^\vee$ satisfies $m_{i-1}^\phi = (m_{i-1}s)^\theta = 1$. In other words, ϕ is the generator of $M_{1-i}^\vee$ and $\phi \in S_1 M_i^\vee$. This proves the claim.

By the claim $M^\vee = SM_{1-r}^\vee$ and so it is the shift of a truncated left point module. By hypothesis (ii), $M^\vee$ is a homomorphic image of a unique shifted point module and so there exists a unique shifted truncated point module $L = \bigoplus_{i=1-r}^{1} L_i$ such

that $M^\vee = L/L_1$. Taking Matlis duals, again, gives the required module $M' = L^\vee$. The uniqueness of L implies the uniqueness of M'.

(2) We prove the statement for Y^r only; the proof for Y^ℓ is symmetric. For each finitely generated commutative k-algebra C, in part 1 we proved that s induces a bijection from the set $\mathcal{S}_{\mathrm{pt}}(C)$ to itself. It is easy to check that these bijections are functorial, so for any commutative k-algebra C, we get an induced bijection from $\mathcal{S}_{\mathrm{pt}}(C)$ to itself by taking limits over the finitely generated subalgebras C' of C. Thus we have actually defined a natural isomorphism from the functor $\mathcal{S}_{\mathrm{pt}}$ to itself in the category (Fun). Since Yoneda's lemma embeds (Schemes) as a full subcategory of (Fun), and $\mathcal{S}_{\mathrm{pt}}$ is represented by the scheme Y^r, we must have a scheme automorphism $\sigma : Y^r \to Y^r$ induced by s.

(3) Fix a commutative noetherian ring C and $\mathcal{M} = \pi(M) \in \mathcal{S}_{\text{q-pt}}(C)$. As S_C is generated in degree one, we may choose a tail $N = M_{\geq n} = \bigoplus_{i \geq n} M_i$ of M such that $N = N_n S_C$ and N_i is a projective C-module locally of rank one for all $i \geq n$. By part 1, N is the tail $L_{\geq n}$ of a unique point module $L \in \mathcal{S}_{\mathrm{pt}}(C)$. Thus the sets $\mathcal{S}_{\mathrm{pt}}(C)$ and $\mathcal{S}_{\text{q-pt}}(C)$ are in natural bijection for all finitely generated k-algebras C and the same holds for all $C \in$ (Rings) by taking limits. Thus $\mathcal{S}_{\mathrm{pt}}(C)$ and $\mathcal{S}_{\text{q-pt}}(C)$ are naturally isomorphic functors and $\mathcal{S}_{\text{q-pt}}$ is also represented by Y^r. $\square$

We now turn to the structure of the point modules over $R = R(X, c, \mathcal{L}, \sigma)$ and show that R satisfies none of the conclusions of Theorem 10.1 or Proposition 10.2. To do this we either have to assume that R is generated in degree one (since that is required for the definition of point modules) or to work with general R and use a slightly more artificial class of R-modules. We will use the second approach in a way that also includes the first case.

Notation 10.3. Fix an open affine subset $U \subset X$ and recall from Theorem 9.2 that $\mathcal{R}(U)$ is a finitely generated right $R_{\mathcal{O}_X(U)}$-module. Fix a finitely generated graded free R-module P such that $\mathcal{R}(U)$ is a homomorphic image of $P_{\mathcal{O}_X(U)} = P \otimes_R R_{\mathcal{O}_X(U)}$. Now take h to be the constant function 1 and let $\mathcal{P} = \mathcal{P}_h$ be the corresponding functor.

We would like to thank Brian Conrad for his help which was invaluable in the proof of the next result.

Theorem 10.4. *Keep the hypotheses of* (4.9) *and* (10.3)*. Then:*

1. *The functor $\mathcal{P}$ is not represented by any scheme Y of locally finite type.*
2. *For $m \gg 0$, set $S = R(X, c, \mathcal{L}^m, \sigma)$. Then S is generated in degree one but $\mathcal{S}_{\mathrm{pt}}$ is not represented by a scheme Y of locally finite type.*

Proof. (1) Assume that $\mathcal{P}$ is represented by the scheme Y of locally finite type. The intuitive reason for our choice of $\mathcal{P}$ is that $\mathcal{R}(U)$ is "almost" in $\mathcal{P}(\mathcal{O}_X(U))$. More precisely, the choice of P ensures that $\mathcal{R}(U)$ is a homomorphic image of $P_{\mathcal{O}(U)}$. Moreover, if $p \in U \smallsetminus \{c_i\}$ is a closed point then $(\mathcal{I}_n \otimes_{\mathcal{O}_X} \mathcal{L}_n)_p \cong \mathcal{O}_{X,p}$, for all $n \geq 0$. Thus, if $C = \mathcal{O}_{X,p}$ for some such p, then $\mathcal{R}_p = \mathcal{R}(U) \otimes_{\mathcal{O}(U)} C \cong \bigoplus_{i \geq 0} C$ does belong to $\mathcal{P}(C)$. However, Theorem 9.2 and Remark 9.3(1) imply that $\mathcal{R}(U)$ is not a flat $\mathcal{O}(U)$-module; indeed it fails to be flat at precisely the points in $U \cap \{c_i\}$. Thus, $\mathcal{R}(U)$ does not lie in $\mathcal{P}(\mathcal{O}(U))$.

Fix a closed point $p \in U \smallsetminus \{c_i : i \in \mathbb{Z}\}$ and set $C = \mathcal{O}_{X,p}$. The last paragraph implies that there exists $\theta_p \in \mathcal{P}(C) = \mathrm{Morph}(\operatorname{Spec} C, Y)$ corresponding to $\mathcal{R}_p$. By the definition of locally finite type [Ha, p.84], we may pick an open affine

neighbourhood V of $\theta_p(p)$ in Y of finite type over k. Then we get a map of algebras $\theta'_p : \mathcal{O}_Y(V) \to \mathcal{O}_{\operatorname{Spec} C}(\theta_p^{-1}(V))$. Since $\theta_p^{-1}(V)$ is an open set containing p, it is necessarily $\operatorname{Spec} C$ and so $\operatorname{Im}(\theta'_p) \subseteq C$. Since $\mathcal{O}_Y(V)$ is a finitely generated k-algebra and $\mathcal{O}_X(U)$ is a domain, $\theta'_p(\mathcal{O}_Y(V)) \subseteq \mathcal{O}_X(U')$, for some open set $U' \subseteq U$. Since it does no harm to replace U by a smaller open set containing p, we may as well assume that $U = U'$. In other words, we have extended θ_p to a map $\widetilde{\theta}_p \in \operatorname{Morph}(U, Y)$ such that $\theta_p = \widetilde{\theta}_p \circ \pi_p$, where $\pi_p : \operatorname{Spec} C \to U$ is the natural morphism.

By construction, $\widetilde{\theta}_p$ corresponds to a module $M_U \in \mathcal{P}(\mathcal{O}(U))$ with the property that $M_U \otimes_{\mathcal{O}(U)} C \cong \mathcal{R}_p$. But $\mathcal{R}(U)$ is a second finitely generated $R_{\mathcal{O}(U)}$-module with $\mathcal{R}(U) \otimes_{\mathcal{O}(U)} C \cong \mathcal{R}_p$. This local isomorphism of R_C-modules lifts to an isomorphism $M_W = M_U \otimes_{\mathcal{O}(U)} \mathcal{O}(W) \cong \mathcal{R}(W)$ of $R_{\mathcal{O}(W)}$-modules, for some open affine set $W \subseteq U$. By the definition of $\mathcal{P}$, $(M_W)_n = (M_U)_n \otimes_{\mathcal{O}(U)} \mathcal{O}(W)$ is a flat $\mathcal{O}(W)$-module for all n. On the other hand, for $n \gg 0$, Remark 9.3(1) implies that $(M_W)_n \cong \mathcal{R}(W)_n$ is *not* flat over $\mathcal{O}(W)$. This contradiction proves (1).

(2) Pick an integer M by Proposition 4.12 and assume that $m \geq M$. Then S is generated in degree one and $P = S$ satisfies the hypotheses of (10.3). Thus, part 2 follows from part 1. □

With minor changes the proof of Theorem 10.4 also shows that the point modules in qgr-R are not parametrized by a scheme of locally finite type.

Corollary 10.5. *Keep the hypotheses of* (4.9) *and* (10.3). *Then the functor* $\mathcal{P}^{\text{qgr}} = \mathcal{P}_1^{\text{qgr}}$ *is not represented by any scheme* Y *of locally finite type.*

Similarly, if $S = R(X, c, \mathcal{L}^m, \sigma)$ *for* $m \gg 0$, *then* S *is generated in degree one but* $\mathcal{S}_{\text{q-pt}}$ *is not represented by a scheme* Y *of locally finite type.*

Remark 10.6. Combined with Theorem 10.4 and Remark 10.8, this proves Theorem 1.1(5).

Proof. Consider the proof of Theorem 10.4(1). In the final paragraph, $M_W \in \mathcal{P}(\mathcal{O}(W))$ and so it certainly lies in $\mathcal{P}^{\text{qgr}}(\mathcal{O}(W))$. However, as $\mathcal{R}(W)_n$ is not flat as an $\mathcal{O}(W)$-module for any $n \gg 0$, no tail $\mathcal{R}(W)_{\geq n}$ of $\mathcal{R}(W)$ is a flat $\mathcal{O}(W)$-module and hence $\mathcal{R}(W)$ cannot belong to $\mathcal{P}^{\text{qgr}}(\mathcal{O}(W))$.

Thus, the proof of Theorem 10.4 can also be used to prove the corollary. □

Corollary 10.5 is in stark contrast to Remark 6.8. To see this, assume that $R = S$, so that the functor $\mathcal{P} = \mathcal{S}_{\text{q-pt}}$ determines point modules. It follows from Remark 10.8 below that the points in qgr-R are simply the images of the point modules in gr-R. Thus Remark 6.8 can be rephrased as saying that, in qgr-R, the point modules are in (1-1) correspondence with the points of X—which is definitely a scheme of finite type. The way to think of the difference is as follows: If the point modules in qgr were indeed parametrized by X then, in qgr-$R_{\mathcal{O}(U)}$, one would need to find not only the point modules induced from $R = R_k$ but also a module corresponding to the immersion $U \subseteq X$. The proof of Corollary 10.5 can be interpreted as saying that one does have indeed a module corresponding to U. Unfortunately it is $\pi(\mathcal{R}(U))$ which, as the proof also shows, is not a point module in qgr-$R_{\mathcal{O}(U)}$.

Although the last few results have shown that one cannot parametrize the point modules for R, they do not say anything about the point modules over $R = R_k$ itself. The next two results consider these modules and show that they also have

interesting properties. For any closed point $x \in X$, recall the definition of the modules $\widetilde{x} = \bigoplus_{i\geq 0} k(x)_{\sigma^i} \in \text{qgr-}\mathcal{R}$ from (5.2).

Proposition 10.7. *Keep the hypotheses of* (4.9). *Set* $\dim X = d$ *and write* $R = R(X, c, \mathcal{L}, \sigma)$. *If* x *is a closed point in* X, *then the* R*-module of global sections* $\Gamma_{AZ}(\widetilde{x})$ *has Hilbert series*

$$H(t) = \begin{cases} \sum_{i=0}^{r} dt^i + \sum_{i=r+1}^{\infty} t^i & \textit{if } \ x = c_r \ \textit{ for } \ r \geq 0; \\ 1/(1-t) & \textit{otherwise.} \end{cases}$$

Remark 10.8. By Remark 6.8 and the equivalence of categories Theorem 4.1, the simple objects in Qgr-$\mathcal{R} \simeq$ Qgr-R are precisely $\{\widetilde{x} : x \text{ a closed point in } X\}$. Thus, if R is generated in degree one, the proposition implies that these objects are precisely the point modules in qgr-R.

Proof. We remind the reader that Γ_{AZ} denotes the Artin-Zhang sections functor from (7.10). By Lemma 5.5, $\widetilde{x}[m] = \bigoplus_i (k(x)^{\sigma^{-m}})_{\sigma^i} = \bigoplus_i k(\sigma^m(x))_{\sigma^i}$. By Lemma 6.4(1) and Theorem 4.1,

$$\Gamma_{AZ}(\widetilde{x})_m = \lim_{n\to\infty} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_n,\, k(\sigma^m(x))) \qquad \text{for } m \geq 0.$$

There are two possibilities. If $x = c_j$ with $0 \leq j-m \leq n-1$, then $\sigma^m(x) = c_{j-m}$ and $\operatorname{Hom}(\mathcal{I}_n, k(c_{j-m})) \cong \Gamma(\mathcal{I}_{c_{j-m}}/\mathcal{I}^2_{c_{j-m}}) \cong k^d$. For all other choices of x one has $\operatorname{Hom}(\mathcal{I}_n, k(x)) = k$. In other words,

$$\Gamma_{AZ}(\widetilde{x})_m = \begin{cases} k^d & \text{if } \ x = c_j \ \text{ with } \ j \geq m; \\ k & \text{otherwise.} \end{cases}$$

This is equivalent to the assertion of the proposition. □

Corollary 10.9. *Assume that* $R = R(X, c, \mathcal{L}, \sigma)$ *is generated in degree one. Then the shift functor* $s : P \mapsto P[1]_{\geq 0}$ *does not induce an automorphism on the set of isomorphism classes of* R*-point modules.*

Proof. Set $M = \Gamma_{AZ}(\widetilde{c}_0)$. By Proposition 10.7, $\dim_k M_0 = d \geq 2$ but $\dim M_j = 1$ for $j > 0$. Pick two linearly independent elements $\alpha_i \in M_0$ and write $M^i = \alpha_i R$. We will show that the M^i are non-isomorphic point modules for R.

By [AZ1, S2, p.252], M and hence the M^i are torsion-free, in the sense that they have no finite dimensional submodules. Since R is generated in degree one and $\dim_k M_i = 1$, for $i > 0$, this forces $(M^i)_{\geq 1} = M_{\geq 1}$ and so certainly the M^i are point modules. Suppose that there is an isomorphism $\theta : M^1 \to M^2$. Adjusting θ by a scalar, we may suppose that $\theta(\alpha_1) = \alpha_2$. Now, $(M^1)_{\geq 1} = (M^2)_{\geq 1}$ is a shifted point module and so its only automorphisms are given by scalar multiplication. Hence the restriction of θ to $(M^1)_{\geq 1}$ is given by multiplication by such a scalar; say λ. Thus, for all $r \in R_1$, $\lambda\alpha_1 r = \theta(\alpha_1 r) = \alpha_2 r$. In other words, $(\lambda\alpha_1 - \alpha_2)R_1 = 0$, contradicting the fact that M is torsion-free. Thus the M^i are nonisomorphic. Since $s(M^1) = (M^1)_{\geq 1}[1] = s(M^2)$ this implies that the shift functor s is not an injection. □

The results of this section give several more ways in which the properties of R differ from those of a strongly noetherian k-algebra. Indeed, if R were strongly noetherian, then Theorem 10.4 would contradict [AZ2, Theorem E4.3], while, if R

were also generated in degree one, then Corollary 10.5 would contradict Proposition 10.2(3) and Corollary 10.9 would contradict Proposition 10.2(1).

11. Examples

In this last section we discuss the stringency of Assumptions 4.9 and give a number of examples where the hypotheses are satisfied. As is shown in [Ke1], projective schemes X with σ-ample sheaves $\mathcal{L}$ exist in abundance. So the main issue is to determine the varieties X and automorphisms σ for which there exists a critically dense orbit $\mathcal{C} = \{\sigma^i(c)\}_{i\in\mathbb{Z}}$.

When $X = \mathbb{P}^t$, [Ro, Section 14] shows that $\mathcal{C}$ is critically dense for generic choices of σ and c. Below, we provide further examples of varieties for which $\mathcal{C}$ is critically dense for many choices of σ, c. The main technique is to find situations where one can reduce the problem of proving that $\mathcal{C}$ is critically dense to the problem of proving that $\mathcal{C}$ is dense. For this we use the following theorem of Cutkosky and Srinivas.

Theorem 11.1. [CS, Theorem 7] *Let G be a connected commutative algebraic group defined over an algebraically closed field k of characteristic 0. Suppose that $g \in G$ is such that the cyclic subgroup $H = \langle g\rangle$ is dense in G. Then any infinite subset of H is dense in G.* □

Now let X be any integral projective scheme with automorphism σ and closed point c. We think of X as a variety in this section.

Theorem 11.2. *Assume that $\operatorname{char} k = 0$. Suppose that there is an algebraic group $G \subseteq \operatorname{Aut} X$ with $\sigma \in G$ such that the map $\theta : G \times X \to X$ defined by $(g, x) \mapsto gx$ is regular. Then $\mathcal{C} = \{\sigma^i(c) : i \in \mathbb{Z}\}$ is dense in X if and only if $\mathcal{C}$ is critically dense in X.*

Proof. Assume that $\mathcal{C}$ is dense in X. Let Z be the closure in G of the subgroup $H = \langle\sigma\rangle$ of G. Then Z is a subgroup of G [Hu, Proposition 7.4.A], which is abelian since H is.

By definition, an algebraic group is of finite type, so Z has finitely many connected components; let Z_0 be the connected component containing 1. Since $\mathcal{C}$ is dense in X, the points $c_i = \sigma^{-i}(c)$ are distinct (unless X is a point, in which case the theorem is trivial) and so the automorphisms $\{\sigma^i\}_{i\in\mathbb{Z}}$ must be distinct points in $\operatorname{Aut} X$. Thus $H \cong \mathbb{Z}$ as groups and it is easy to see that Z_0 is the closure of $H_0 = \langle\sigma^e\rangle$ for some $e \geq 1$. Then the connected components of Z are the closures Z_j of the cosets $H_j = \sigma^j H_0$ for $0 \leq j \leq e-1$. Now apply Theorem 11.1 to $Z_0 \supseteq H_0$ to show that H_0 is critically dense in Z_0 and hence that each $H_j = \sigma^j H_0$ is critically dense in $Z_j = \sigma^j Z_0$.

If $\mathcal{C}$ is not critically dense, pick $\{c_i\}_{i\in I} \subseteq W \subsetneq X$, where W is closed and I is infinite. We can choose some $0 \leq j \leq e-1$ for which $I' = I \cap \{-j + e\mathbb{Z}\}$ is still infinite. Now let $\phi : Z_j \times \{c\} \to X$ be the restriction of θ, and let $\rho_1 : Z_j \times \{c\} \to Z_j$ be the first projection. Then $\{\sigma^{-i}\}_{i\in I'} \subseteq \rho_1(\phi^{-1}(W)) \subseteq Z_j$. Since H_j is critically dense in Z_j, we must have $\rho_1(\phi^{-1}(W)) = Z_j$. But then $\{c_i \mid i \in -j + e\mathbb{Z}\} = \phi\rho_1^{-1}(H_j) \subseteq W$, and so $\mathcal{C} \subseteq \bigcup_{i=0}^{e-1} \sigma^i(W) \subsetneq X$, since X is irreducible. This contradicts the density of $\mathcal{C}$ in X and shows that $\mathcal{C}$ is critically dense. The other direction is trivial. □

If σ, τ are automorphisms of projective schemes X and Y, respectively, and the hypotheses of Theorem 11.2 are satisfied for $\sigma \in G \subseteq \operatorname{Aut} X$ and $\tau \in G' \subseteq \operatorname{Aut} Y$, then those hypotheses are also satisfied for $(\sigma, \tau) \in G \times G' \subseteq \operatorname{Aut}(X \times Y)$. As an application of this remark, we will find critically dense sets in products of projective spaces. Here we think of elements of $\mathrm{PGL}(t+1) = \operatorname{Aut} \mathbb{P}^t$ as $(t+1) \times (t+1)$ matrices acting by left multiplication on points of $\mathbb{P}^t$ written as column vectors of homogeneous coordinates.

Example 11.3. *Assume that* $\operatorname{char} k = 0$ *and let* $X = \mathbb{P}^{s_1} \times \mathbb{P}^{s_2}$. *Let* $\tau_i \in \operatorname{Aut}(\mathbb{P}^{s_i})$ *be given by the diagonal matrix* $\operatorname{diag}(1, p_{i1}, \dots, p_{is_i})$. *Assume that the* $\{p_{ij}\}$ *generate a multiplicative subgroup of* k *isomorphic to* $\mathbb{Z}^{s_1+s_2}$ *and write* $\sigma = (\tau_1, \tau_2)$.

If $c = ((1:1:\dots:1),(1:1:\dots:1)) \in X$, *then* $\mathcal{C} = \{\sigma^i(c) : i \in \mathbb{Z}\}$ *is critically dense in* X. *Moreover, if* $\mathcal{L}$ *is any very ample invertible sheaf on* X *then* $\mathcal{L}$ *is* σ*-ample and so* (4.9) *does hold for the data* $(X, c, \mathcal{L}, \sigma)$.

Proof. Since $\sigma \in \mathrm{PGL}(s_1+1) \times \mathrm{PGL}(s_2+1) \subseteq \operatorname{Aut} X$, by Theorem 11.2 it is enough to prove that $\mathcal{C}$ is dense in X. If this is false, there exists a proper closed set Y with $\mathcal{C} \subseteq Y \subsetneq X$ and $\sigma(Y) = Y$. Let I be the defining ideal of Y in the bigraded polynomial ring $U = k[x_{10}, x_{11}, \dots, x_{1s_1}, x_{20}, x_{21}, \dots, x_{2s_2}]$. Thus I is bihomogeneous in the x's and satisfies $\phi(I) = I$, where ϕ is the automorphism of U corresponding to σ; explicitly, ϕ is defined by $\phi(x_{ij}) = p_{ij}x_{ij}$. Each nonzero bihomogeneous component I_{uv} of I is fixed by ϕ and so contains an eigenvector, say $f = f_{uv}$, for ϕ. But the hypotheses on the p_{ij} then force f to be a single monomial in the x's, and so $f(c) \neq 0$, contradicting $c \in Y$. Thus $\mathcal{C}$ is dense in X.

The canonical sheaf on $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(-n-1)$ which is certainly minus ample. By [Ha, Exercises II.8.3 and II.5.11], the canonical sheaf on $\mathbb{P}^{s_1} \times \mathbb{P}^{s_2}$ is therefore also minus ample. Now apply [Ke1, Proposition 5.6] to see that $\mathcal{L}$ is σ-ample. $\square$

Another large class of examples is provided by abelian varieties. Here we recall some relevant definitions and refer the reader to [Ln] for the details. An abelian variety E is called *simple* if the only (irreducible) abelian subvarieties of E are itself and 0. Two abelian varieties E, E' are called *isogenous* if there is a surjective morphism of abelian varieties $E \to E'$ with finite kernel. This is an equivalence relation on the set of abelian varieties and every abelian variety E is isogenous to a finite product of simple abelian varieties [Ln, Corollary, p.30].

The following result is proved in [RZ].

Proposition 11.4. *Let* $E = E_1 \times E_2 \times \dots \times E_n$, *where the* E_i *are simple abelian varieties. Let* $a = (a_1, a_2, \dots, a_n) \in E$, *and let* Z_a *be the Zariski closure in* E *of* $\{ia\}_{i \in \mathbb{Z}}$. *Then:*

(1) *There is a countable set of closed subsets* $Y_\alpha \subsetneq E$ *such that* $Z_a = E$ *for all* $a \notin \bigcup Y_\alpha$.
(2) *If the* E_i *are pairwise non-isogenous, then* $Z_a = E$ *if and only if each* a_i *is a point of infinite order in* E_i. $\square$

If k is an uncountable field, part 1 of the proposition shows that, for a sufficiently general point $a \in E$, we have $Z_a = E$. Part 2 shows that in the special case where E is a product of non-isogenous simples it is easy to describe exactly for which $a \in E$ this happens. Thus the hypotheses of the next result hold for generic $a \in E$.

Example 11.5. *Let E, a, and Z_a be as in Proposition 11.4. Assume that E is defined over a field of characteristic zero and that $Z_a = E$. Let $\sigma : E \to E$ be the translation automorphism defined by $x \mapsto x + a$ and pick any $c \in E$. Then $\mathcal{C} = \{\sigma^i(c)\}_{i\in\mathbb{Z}}$ is critically dense in E. If $\mathcal{L}$ is a very ample invertible sheaf on E then $\mathcal{L}$ is σ-ample and so* (4.9) *does hold for the data* $(E, c, \mathcal{L}, \sigma)$.

Proof. The group of all translation automorphisms of E is isomorphic to E, so it is an algebraic subgroup of $\operatorname{Aut} E$ containing σ. By Theorem 11.2 we just need to show that $\mathcal{C}$ is dense. This clearly does not depend on the choice of c, so we may choose $c = 0$. By hypothesis, the closure of $\mathcal{C}$ is $Z_a = E$, as required. The final assertion is proved in [RZ]. □

When k is a field of characteristic $p > 0$, there do exist examples of orbits $\mathcal{C} = \{\sigma^i(c_0)\}$ that are dense but not critically dense [Ro, Example 14.9]. However, we cannot answer:

Question 11.6. *If* $\operatorname{char} k = 0$*, is every dense orbit* $\{\sigma^i(c) : i \in \mathbb{Z}\}$ *critically dense?*

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